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Mathematical Tables
and other
Aids to Computation

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Note on the Computation of the Bessel Function $I_n(x)$.

The Bessel function of purely imaginary argument

$$(1) \qquad I_n(x) = i^{-n}J_n(ix) = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{\Gamma(m+1)\Gamma(m+n+1)}$$

is of frequent occurrence in applied mathematics, statistics, and asymptotic number theory. The calculation of $I_n(x)$ for n and x both large presents an interesting problem to the computer.[1] It is clear that the series for $I_n(x)$, (though everywhere convergent) is of little practical use for large x. Even the well-known asymptotic formula, designed for use with large x

$$(2) \quad I_n(x) = \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\left(1 - \frac{4n^2 - 1^2}{1!\,8x} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2!\,(8x)^2} - \cdots\right) + O(e^{-x}x^{-\frac{1}{2}})$$

fails to be of practical use when n is of order $\sqrt{x}$.

Bessel functions of large integral order can, theoretically, be built up from the values of $I_0(x)$ and $I_1(x)$ (given in tables,[2] or found by (2)) by use of the recurrence formula

$$(3) \qquad I_{k+1}(x) = -2kI_k(x)/x + I_{k-1}(x).$$

This procedure, however, affords another example in which great accuracy is needed to start with (as has been pointed out in case of $J_n(x)$ by COSENS [*MTAC* p. 99]), since more and more accuracy is lost at each step. The value of $I_{90}(60) = 0.0725775\cdots$ would, if computed this way, require about 40 decimal places in $I_0(60)$ and $I_1(60)$, and 89 applications of (3).

NICHOLSON[3] has given a formula for $I_n(x)$ purporting to be "suitable for rapid tabulation," when either n or x is large. The formula is not given explicitly (except its first term) but in terms of an infinite differential operator with undetermined coefficients operating on a cumbersome function of n and x.

The following formula, which I have not found in the literature, is quite effective in case n is large and x has any real value.

$$(4) \qquad I_n(nx) = \frac{1}{(2\pi nz)^{\frac{1}{2}}}\{(z-1)e^z/x\}^n \exp \sum_{m=1} \psi_m(x^2)/(nz^2)^m$$

where $z = (1 + x^2)^{\frac{1}{2}}$, and where the ψ's are the following polynomials:

$$\begin{aligned}
24\,\psi_1(x^2) &= 3x^2 - 2,\\
16\,\psi_2(x^2) &= x^4 - 4x^2,\\
5760\,\psi_3(x^2) &= 375x^6 - 3654x^4 + 1512x^2 - 16,\\
128\,\psi_4(x^2) &= 13x^8 - 232x^6 + 288x^4 - 32x^2,\\
322560\,\psi_5(x^2) &= 67599x^{10} - 1914210x^8 + 4744640x^6 - 1891200x^4\\
&\qquad + 78720x^2 - 256,\\
192\,\psi_6(x^2) &= 103x^{12} - 4242x^{10} + 17493x^8 - 14884x^6 + 2580x^4 - 48x^2,\\
3440640\,\psi_7(x^2) &= 5635995x^{14} - 318291750x^{12} + 1965889800x^{10}\\
&\qquad - 2884531440x^8 + 1135145088x^6 - 99783936x^4\\
&\qquad + 881664x^2 - 2048,\\
4096\,\psi_8(x^2) &= 23797x^{16} - 1765936x^{14} + 15252048x^{12} - 34280896x^{10}\\
&\qquad + 24059968x^8 - 5095936x^6 + 248320x^4 - 1024x^2.
\end{aligned}$$

This formula may be derived from an asymptotic expansion of the solution

$$y = d(\ln I_n(nx))/dx$$

of the differential equation

$$y' + y/x + y^2 = n^2(1 + x^2)/x^2 = (nz/x)^2.$$

Further polynomials $\psi_m(t)$, if needed, may be found expeditiously as follows. Define the polynomial $Q_m(t)$ by

$$Q_0(t) = 1, \qquad Q_1(t) = 5 - t, \qquad Q_2(t) = 60 - 48t + 4t^2,$$

and in general by the recurrence formula [4]

$$Q_m = (6m - (6m - 4)t)Q_{m-1} - 4t(1 - t)Q'_{m-1} + (1 - t)\sum_{\lambda=2}^{m-1} Q_{\lambda-1}Q_{m-\lambda}.$$

Then the mth term T_m of the asymptotic series occurring in (4) may be written

$$T_m = \tfrac{1}{2}(-4n)^{-m}\int_z^\infty Q_m(t^2)t^{-2m-1}dt.$$

In other words, if we denote the coefficient of t^k in $Q_m(t)$ by $A_k^{(m)}$ and define the polynomial $\phi_m(t)$ by

$$\phi_m(t) = \sum_{k=0}^{m} A_k^{(m)}t^k/(3m - 2k),$$

then

$$T_m = (-1)^m 2^{-2m-1}n^{-m}z^{-3m}\phi_m(z^2).$$

As a matter of fact, the polynomial

$$\psi_m(t) = \phi_m(1 + t)$$

has simpler coefficients than $\phi_m(t)$ and since $\phi_m(z^2) = \psi_m(x^2)$ we have given T_m partly in terms of x. In doing this we are following MEISSEL[5] whose formulas for $J_n(nx)$ also involve $\psi_m(t)$ for $m \leqq 6$. A comparison of his extended results reveals four errors in his $\psi_6(\sec^2\alpha)$. *For* denominators

1, 4, 192, 256, 128, 768, *read* 1, 4, 48, 16, 8, 48.

This incorrect polynomial has produced an error in the corresponding term in the formula as quoted by WATSON[6]

for 3072, 768, 41280, *read* 192, 48, 2580.

WATSON has another error (not due to MEISSEL) in the polynomial where 98720 is a misprint for 78720; noted by W. G. BICKLEY, *Phil. Mag.*, s. 7, v. 34, 1943, p. 45.

To illustrate the behavior of (4) for a moderate value of x we take $x = \frac{3}{4}$ so that $z = \frac{5}{4}$. In this case (4) becomes:

$$\begin{aligned}\ln I_n(3n/4) = (\tfrac{5}{4} - \ln 3)n - \tfrac{1}{2}\ln n - \tfrac{1}{2}\ln(5\pi/2) - 1/150n - 99/3125n^2 \\ - 1044097/175781250n^3 + 868626/48828125n^4 \\ + 857735827/76904296875n^5 \\ - 4925585088/152587890625n^6 \\ + 4590654771247901/100135803222656250n^7 \\ + 278080398059472/2384185791015625n^8 + \cdots.\end{aligned}$$

For $n = 100$ these various terms are, to 20 decimals,

$$\begin{aligned}
&15.13877\ 11331\ 89030\ 86048\\
-\ &2.30258\ 50929\ 94045\ 68402\\
-\ &1.03051\ 03088\ 84078\ 40331\\
-\ &.00006\ 66666\ 66666\ 66667\\
-\ &.00000\ 31680\ 00000\ 00000\\
-\ &.00000\ 00059\ 39751\ 82222\\
+\ &.00000\ 00001\ 77894\ 60480\\
+\ &.00000\ 00000\ 01115\ 32887\\
-\ &.00000\ 00000\ 00032\ 28031\\
-\ &.00000\ 00000\ 00000\ 45844\\
+\ &.00000\ 00000\ 00000\ 01166,
\end{aligned}$$

and their sum is

$$11.80560\ 58908\ 83465\ 49\cdots.$$

Thus

$$I_{100}(75) = 134001.44880\ 18810.$$

D. H. L.

[1] Such Bessel functions have been used recently at the statistical laboratory of the University of California in preparation of certain statistical tables to appear in *Annals of Math. Statistics*. See also: J. WISHART, "A note on the distribution of the correlation ratio," *Biometrika*, v. 24, 1932, p. 454, formula (27).

[2] The most extensive tables of $I_n(x)$ are in B.A.A.S., *Math. Tables*, v. 6, *Bessel Functions*, part I, Cambridge, 1937, Tables VI and VIII.

[3] J. W. NICHOLSON, *Phil. Mag.* s. 6, v. 20, 1910, p. 938–943.

[4] The Q's may be checked by the relation $Q_m(1) = 4^m$; also $m(m+1)\int_0^1 t^{2m-1}Q_m(t^{-2})dt = 2^{2m+1}|B_{m+1}|$, where B_k is the kth Bernoulli number, in the notation of Lucas.

[5] D. F. E. MEISSEL, *Astr. Nach.* v. 130, 1892, cols. 363–4.

[6] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge 1922, p. 228.

Mathematical Tables in Phil. Mag.

The important scientific periodical *London, Edinburgh & Dublin Philosophical Magazine and Journal of Science*, has been published under different titles since 1798, in 7 series of volumes (varying in number from 11 to 68). In recent years the usual print abbreviation of the title of this periodical has been *Phil. Mag.* In what follows it is proposed to list, with a few added notes, mathematical tables in *Phil. Mag.*, s. 3, v. 18 to s. 7, v. 34, no. 235, inclusive, 1841–1943. It will be observed that more than half of the Tables listed were published after the last world war started in 1914, a period of unparalleled scientific development. Many mathematical tables have been already published as the result of problems arising in the prosecution of the present war; the number is likely to be greatly increased in the next few years.

S. 3

1. G. B. AIRY "On diffraction of an annular aperture," v. 18, 1841, p. 7. Table of $E[= J_0(e)] = (1/2\pi)\int_0^{2\pi}\cos(e\cos\theta)d\theta$, and of E^2, for $e = [0.0(0.2)10.0; 4D]$. Airy had an earlier table of $2J_1(e)/e$ of the same scope in Cambridge Phil. So., *Trans.*, v. 5, 1835, p. 291. In B.A.A.S., *Math. Tables*, v. 6, *Bessel Functions*, Cambridge, 1937, J_0 is given, among other values, for $e = [0.0(0.2)25.0; 8D]$. The value of $J_0(40)$, to 7D, was computed by W. R. Hamilton, *Phil. Mag.*, s. 4, v. 14, 1857, p. 381.

2. J. W. LUBBOCK, "On the determination of the numerical values of the coefficients in any series consisting of sines and cosines of multiples of a variable angle," v. 33, 1848, p. 106–127. "In the following methods of developing any function of sines and cosines by means of particular values of the function, the artifice by which the coefficients are successively eliminated is taken from Leverrier's 'Développements sur plusieurs points de la théorie des perturbations des planètes.' " Lubbock gives various tables for evaluating numerical constants when the method is practically employed.

S. 4

3. J. W. L. GLAISHER, "On a class of definite integrals—Part II," v. 42, 1871, p. 431–436, "Tables of the error function." Values of $\int_x^\infty e^{-u^2}du$ are given for $x = [3.00(0.01)4.50; 7S]$. The range $x = [0.00(0.01)3.00; 8D$ to $11D]$ was covered in a table, with first three differences, in C. KRAMP, *Analyse des Réfractions Astronomiques et Terrestres*, Leipzig and Paris, 1799, p. 195–202. In A. A. MARKOV, *Table des Valeurs de l'Intégrale* $\int_x^\infty e^{-t^2}dt$, St. Petersburg, 1888, $x = [0.000(0.001)3.00(0.01)4.80; 11D]$, and first, second, and third differences are given.

S. 5

4. H. M. TAYLOR, "On the relative values of the pieces in chess," v. 1, 1876, p. 229. Table of "chance of the king being in check" for chessboard of n^2 squares (n even).

5. J. W. STRUTT, *baron* RAYLEIGH, "Investigations in optics, with special reference to the spectroscope," v. 8, 1879, p. 266–268. For varying values of u, $u^{-3}\sin^2 u$, to $4D$, is given in T. I; $f = u^{-1}\left(\sin u - \sin\frac{u}{6}\right)$, as well as $(f/f_0)^2$, are given to $4D$ in T. II; and $f = u^{-1}\left(\sin u - 2\sin\frac{u}{8}\cos\frac{u}{3}\right)$, as well as $(f/f_0)^2$ are given in T. III. Also in *Sci. Papers of John William Strutt*, v. 1, Cambridge, 1899, p. 419–422.

6. G. B. AIRY and W. ELLIS, "On a systematic interruption in the order of numerical values of vulgar fractions when arranged in a series of consecutive magnitudes," v. 12, 1881, p. 175–178. Tabular selections from their "Logarithms of the values of all vulgar fractions, with numerator and denominator not exceeding 100, arranged in order of magnitude," Institution of Civil Engineers, *Minutes of Proc.*, v. 65, 1881, p. 271–298; 3043 fractions. Compare no. 7, and RMT 87.

7. J. J. SYLVESTER, "On the number of fractions contained in any 'Farey series' of which the limiting number is given," v. 15, 1883, p. 251–257. A Farey series (see *Phil. Mag.*, s. 1, v. 47, 1816, p. 385) is a system of all the unequal vulgar fractions arranged in order of magnitude, the numerator and denominator of which do not exceed a given number, j. The number of fractions in the series for the limit j, is identical with the sum of the totients of all the natural numbers up to j inclusive—a totient of n, denoted by $\tau(n)$, meaning the number of numbers not exceeding n and prime to it, $T(j) = \sum_{n=1}^{n=j}\tau(n)$. There is a table for $n = 1(1)\ 500$, of $\tau(n)$, $T(n)$, $(3/\pi^2)n^2$. In Sylvester's *Coll. Math. Papers*, v. 4, Cambridge, 1912, p. 103–109 this table is extended to $n = 1000$. For $n = 688$, *for* $\tau(n)$ 536, *read* 336. This error is noted in J. W. L. GLAISHER, *Number-Divisor Tables* (B.A.A.S., *Math. Tables*, v. 8), Cambridge, 1940, where Sylvester's table is extended to $n = 10\,000$. See further D. H. LEHMER, *Guide to Tables in the Theory of Numbers* (Nat. Res. Council *Bull.* no. 105), Washington, D. C. 1941. Compare no. 6.

8. C. E. HOLLAND, P. R. JONES, C. G. LAMB, "Table of zonal spherical harmonics with a short explanation and some illustrations of its use by John Perry." v. 32, 1891, p. 512–523. Also in Phys. So. London, *Proc.*, v. 11, 1892, p. 221–233. This contains the first table of $P_n(\cos\theta)$, $n = [1(1)7; 4D]$, $\theta = 0°(1°)90°$. It was reprinted in W. E. BYERLY, *An Elementary*

Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics . . ., Boston, 1895, p. 278–279. A. H. H. TALLQVIST (Soc. Sci. Fennicae, *Acta*, v. 33, no. 4, 1908) pointed out the numerous and non-trivial errors in this table which Byerly faithfully reproduced except in one case, $P_4(\cos 7°)$, where he correctly put -0.0038 for 0.0038 in the original. Of the 92 errors there are 38 in $P_8(\cos\theta)$ and 48 in $P_7(\cos\theta)$. Of the worst errors samples are as follows: $P_2(\cos 76°)$, *for* -0.4112, *read* -0.4122; $P_6(\cos 7°)$ *for* 0.8476, *read* 0.8492; $P_4(\cos 47°)$ *for* -0.4252, *read* -0.4227; $P_7(\cos 7°)$ *for* 0.7986, *read* 0.8016; $P_7(\cos 18°)$ *for* 0.0289, *read* 0.0248. A correct form of the table, including $P_8(\cos\theta)$, is given in Tallqvist's *Grunderna af Teorin för Sferiska Funktioner . . .*, Helsingfors, 1905, p. 386–387.

9. J. W. STRUTT, *baron* RAYLEIGH, "On the instability of a cylinder of viscous liquid under capillary force," v. 34, 1892, p. 152–153. There is a table of $x^2 + 1 - x^2I_0^2(x)/I_1^2(x) = x^2[J_0^2(ix)/J_1^2(ix)] + 1 + 1/x^2]$, for $x = 0.0, 0.2, 0.4, 0.6, 1.0, 2.0, 4.0, 6.0$, to 3*D* or 4*D*. Also *Scientific Papers*, v. 3, 1902, p. 591–592.

10. H. A. ROWLAND, "Gratings in theory and practice," v. 35, 1893, p. 414. Table of "ghosts" (Bessel functions) J_n^2, for $n = [0(1)14; 3D]$ and varying values of the parameter 0.0 to 10.0.

11. J. PERRY and H. F. HUNT, "The development of arbitrary functions," v. 40, 1895, p. 508–509. Four tables of $xJ_1(x)$ to 4*D*, for varying values of a parameter.

12. H. NAGAOKA, "Diffraction phenomena in the focal plane of a telescope with circular aperture, due to a finite source of light," v. 45, 1898, p. 9. Tables of $I(r) = 1 - J_0^2(r) - J_1^2(r)$ for varying values of r, to 5*D*.

13. B. A. SMITH, "Tables of $\kappa J_0(x) - Y_0(x)$ and $\kappa J_1(x) - Y_1(x)$," v. 45, 1898, p. 122–123. In an article by J. H. Michell on the "wave-resistance of a ship." Tables for $x = [0.00(.01)10.3; 4D]$. $\kappa J_0(x) - Y_0(x) = \int_0^\infty e^{-x\sinh\theta}d\theta - \int_0^{\pi/2}\sin(x\sin\theta)d\theta$. Smith's tables of $Y_0(x)$ and $Y_1(x)$, for $x = [0.00(.01)10.2; 4D]$, are given in *Mess. Math.*, n.s., v. 26, 1896, p. 98–101. This was the earliest table of functions of the second kind.

14. K. PEARSON and A. LEE, "On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling," v. 50, 1900, p. 175. Improbability

$$P = e^{-\frac{1}{2}\chi^2}\left(1 + \frac{\chi^2}{2} + \frac{\chi^4}{2\cdot 4} + \cdots \frac{\chi^{n'-2}}{2\cdot 4\cdot 6\cdots(n'-3)}\right) \quad \text{for } n' \text{ odd;}$$

probability

$$P = (2/\pi)^{\frac{1}{2}}\int_\chi^\infty e^{-\frac{1}{2}\chi^2}d\chi + (2/\pi)^{\frac{1}{2}}e^{-\frac{1}{2}\chi^2}\left(\frac{\chi}{1} + \frac{\chi^3}{1\cdot 3} + \frac{\chi^5}{1\cdot 3\cdot 5} + \cdots \frac{\chi^{n'-2}}{1\cdot 3\cdot 5\cdots(n'-3)}\right)$$

for n' even.

Table of P for $n' = 3(1)20$, $\chi^2 = 1(1)10(5)30(10)70$.

S. 6

15. W. B. MORTON, "On the propagation of polyphase currents along a number of parallel wires," v. 1, 1901, p. 566. $\eta = (\sin\pi/n)^{-2\cos(2q\pi/n)}[\sin(2\pi/n)]^{-2\cos(4q\pi/n)}$. Table of η for $q = 1(1)6$, $n = 2(1)12$.

16. H. HILTON, "A note on van der Waals' equation" $\left[y = \frac{8\theta}{3x-1} - \frac{3}{x^2}\right]$, v. 1, 1901, p. 579–589. Tables I–II for $8\theta = 4, 5, 6, \cdots 16, 24, 32, 40, 48, \cdots 80, -8$, for $x = .4, \cdots, 5.0$; $8\theta = 4, -8$, for $x = .3 \cdots .1, -.5, \cdots -5.00$. Graphs. There are further tables and graphs in Hilton's "A further note on van der Waals' equation," v. 2, 1901, p. 108–118. See also various tables in J. P. Dalton, "On the saturation constants according to van der Waals' equation," v. 13, 1907, p. 517–522.

17. A. A. ROBB, "On the conduction of electricity through gases between parallel plates —part II," v. 10, 1905, p. 672–675. A table gives P, Q, R, S, T, in terms of $\cosh\omega$

$= [1.0(0.1)10;\ 3D]$;

$$P = \frac{4}{5}\cosh^{9/5}\omega \int_\omega^\infty \frac{d\omega}{\cosh^{9/5}\omega}; \quad Q = \cosh^{9/5}\omega; \quad R = \int_0^\infty \left(\frac{9}{5}\cosh^{9/5}\omega \int_\omega^\infty \frac{d\omega}{\cosh^{9/5}\omega}\right) d\omega,$$

$$S = \frac{9}{4}\int_0^\infty \cosh^{9/5}\omega d\omega; \quad T = \frac{S}{Q}\left(\frac{1}{\tanh\omega} - P\right) + R.$$

18. H. G. SAVIDGE, "Tables of the ber and bei and ker and kei functions, with further formulae for their computation," v. 19, 1910, p. 49–58; also in Phys. So. London, *Proc.*, v. 22, 1910, p. 105–114. Tables of ber x, bei x, ker x, kei x, for $x = [1(1)30;\ 4S]$; other tables, p. 54–57, with a similar range. SAVIDGE gave also tables of ker and kei and their first derivatives for $x = [0.1(0.1)10.0;\ 7D–9D]$, in B.A.A.S., *Report*, 1915, p. 36–38.

19. J. R. AIREY, "Tables of Neumann functions $G_n(x)$ and $Y_n(x)$," v. 22, 1911, p. 658–663. Tables of $G_0(x)$, $G_1(x)$, $Y_0(x)$, $Y_1(x)$, for $x = [0.1(0.1)16;\ 7D]$.

20. P. F. WARD, "The transverse vibrations of a rod of varying cross-section," v. 25, 1913, p. 85–106. Various tables corresponding to assumed forms of the rod; Bessel functions basic.

21. J. R. AIREY, "Bessel and Neumann functions of equal order and argument," v. 31, 1916, p. 521–527. The following results are given: first root of $J_{1000}(z) = 0$, to 2D; the values of $J_n(n)$ for $n = 6$ (6D), $n = 48$ (9D), $n = 750$ (14D), $n = 109$ (6D). Also of $J_n(109)$ for $n = [100(1)108;\ 6D]$. Also of $G_n(n)$ and $G_{n-1}(n)$ for $n = [7(1)13;\ 6D]$. And of $G_n(104)$ for $n = [100(1)107;\ 6D]$. See B.A.A.S., *Report*, 1916, p. 92–96 where AIREY has elaborate tables of $J_n(n)$, $J_{n-1}(n)$, $G_n(n)$, $G_{n-1}(n)$, $Y_n(n)$, $Y_{n-1}(n)$ for $n = [1(1)50(5)100(10)200(20)400(50)1000(100)2000(500)5000(1000)20000(5000)30000(10000)\ 50000,\ 100000,\ 500000,\ 1000000;\ 6D]$. Compare nos. **23**, **37**.

22. J. R. AIREY, "The roots of Bessel and Neumann functions of high order," v. 32, 1916, p. 7–14. The first five zeros of $J_{10}(x)$, $J_{100}(x)$, $J_{1000}(x)$, $G_{100}(x)$, and $Y_{100}(x)$ are given to 6S. AIREY's "The roots of the Neumann and Bessel functions," Phys. So. London, *Proc.*, v. 33, 1911, p. 219–24, has the following tables: (a) The first 40 zeros of $Y_0(x)$, $Y_1(x)$, $Y_2(x)$, to 5D; (b) The first 10 zeros of $N_0(x)$, $N_1(x)$, $J_0(x) + Y_0(x)$, $J_1(x) + Y_1(x)$, $J_0(x) - Y_0(x)$, to 5D.

23. G. N. WATSON, "Bessel functions of equal order and argument," v. 35, 1918, p. 369. Incidentally there is a table of $n\int_0^1 J_n(nx)dx$ for $n = [1(2)23;\ 7D]$. Compare no. **21**.

24. H. A. WEBB and J. R. AIREY, "The practical importance of the confluent hypergeometric function," v. 36, 1918, p. 137–141, and plate VI. Table of $M(\alpha, \gamma, x)$ for $\gamma = [1(1)7;\ 4S]$, $\alpha = -3.0(0.5)4.0$, $x = 1(1)6(2)10$.

25. J. R. AIREY, "The addition theorem of the Bessel functions of zero and unit orders," v. 36, 1918, p. 238–242. $J_0(x)$, $G_0(x)$, $Y_0(x)$, $J_1(x)$, $G_1(x)$, $Y_1(x)$ are given for $x = 9$ and 10, to 14D; also auxiliary tables for the computation.

26. J. R. AIREY, "Bessel functions of small fractional order and their application to problems of elastic stability," v. 41, 1921, p. 201–205. First and second zeros of $J_n(x)$ are given for $n = [\frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{1}{4};\ 2D]$ (also the first three zeros for $n = [-\frac{1}{4}, -\frac{3}{4}, -\frac{1}{3}, -\frac{1}{4};\ 2D]$), and $n = [-\frac{1}{2}(0.025)1;\ 4D]$, uncertain fourth decimal places, and the first zeros of $J_{+1/3}(x)$ and $J_{+1/6}(x)$, being also indicated. This last table is reprinted (without indicated uncertainties) in the work of Gray-Mathews-Macrobert, T. XIII, p. 317. Airey gives also the second zero, and $(2n)^{-\frac{1}{3}} \times$ first zero, of $J_n'(x)$ for $n = [-\frac{1}{2}(0.025)\frac{1}{3};\ 4D]$; and $[\frac{1}{4}(n+1)] \times$ the square of the first zero of $J_n(x)$ for $n = [-1(0.025)0;\ 4D]$. Compare no. **27**.

27. A. ONO, "On the first root of Bessel functions of fractional order," v. 42, 1921, p. 1021. With reference to Airey's paper (no. **26**), gives first zeros of $J_{-1/2}$, $J_{-3/7}$, $J_{-1/3}$, J_0, $J_{1/3}$, J_1; 3D.

28. H. E. H. WRINCH and D. M. WRINCH, "Tables of the Bessel functions $I_n(x)$," v. 45, 1923, p. 846–849. Table I: $I_n(x)$, $n = 0(1)6$, $x = [5(1)15;\ 6S]$; T. II: Log $\Gamma(x+1)$ for

$x = [1(1)28; 7D]$; T. III: $(2\pi x)^{\frac{1}{2}}$, e^x, e^{-x}, $e^x/(2\pi x)^{\frac{1}{2}}$, $e^{-x}/(2\pi x)^{\frac{1}{2}}$ for $x = [1(1)20; 6S]$. See also no. **30.**

29. F. F. P. BISACRE, "The calculation of the skin effect in electrical conductors," v. 45, 1923, p. 1035–1038. There are tables (independently checked) for the following functions in $[r, \theta]$ and $[a + ib]$ forms: Table I: $J_0(\sqrt{-i}\,x)$, $J_1(\sqrt{-i}\,x)$; T. II: $H_0^{(2)}(\sqrt{-i}\,x)$, $H_1^{(2)}(\sqrt{-i}\,x)$; for $x = [0.0(0.2)6.0; 4 \text{ or } 5S]$. The tables for the J's in the $[r, \theta]$-form were calculated from the corresponding tables in the $(a + ib)$-form, due to W. S. ALDIS, R. So. London, *Proc.*, v. 66, 1900, p. 42–43. Some columns in Table II were taken from JAHNKE and EMDE, *Funktionentafeln*, Leipzig, 1909, p. 139–140.

30. H. E. H. WRINCH and D. M. WRINCH, "Tables of Bessel functions," v. 47, 1924, p. 62–65. Table I: $I_n(x)$, $n = 0(1)6$, $x = [16(1)37; 6S]$; T. II: $(2\pi x)^{\frac{1}{2}}$, e^x, $e^x/(2\pi x)^{\frac{1}{2}}$ for $x = [21(1)40; 6S]$; T. III–IV: $J_2(x)/I_2(x)$, and $J_3(x)/I_3(x)$, for $x = [1(1)15; 6D]$. The graphs of the functions in Tables III–IV are also given. These functions are of importance in certain problems in acoustics; see R. So. London, *Proc.*, v. 101A., 1922, p. 493–508. See also no. **28.**

31. S. P. OWEN, "Table of values of the integral $\int_0^x K_0(t)dt$," v. 47, 1924, p. 736. Table for $x = [0.02, 0.1, 0.5, 1(1)12; 6S\text{-}7S]$.

S. 7

32. H. E. H. WRINCH and D. M. WRINCH, "The roots of hypergeometric functions with a numerator and four denominators," v. 1, 1926, p. 273–276. First paragraph: "During the last few years, the Generalized Hypergeometric Function has become of increasing importance in applied mathematics. This paper contains certain tables of the roots of certain generalized hypergeometric functions which have important applications in the theory of the vibrations of bars of various cross-sections. Attention was called to the fundamental significance in problems of morphology, of the periods of vibration of bars of varying density with cross-sections of various types, by [A.] DENDY and [J. W.] NICHOLSON [R. So. London, *Proc.*, v. 93A and 89B] in 1917." In this paper are given the first nine roots of the equation $F(\alpha; \alpha_1, \alpha_2, \alpha_3, \alpha_4; -4(x/4)^4) = 0$ for a varied range of values of the parameters.

33. H. P. MULHOLLAND and S. GOLDSTEIN, "The characteristic numbers of the Mathieu equation with purely imaginary parameter," v. 8, 1929, p. 839. Tables are found for the first eight characteristic numbers for purely imaginary q in the Mathieu equation

$$\frac{d^2y}{dx^2} + (4\alpha - 16q\cos 2x)y = 0.$$

34. S. HIGUCHI, "On some closed algebraic curves and their application to dynamical problems," v. 9, 1930, p. 186–190. Tables for solutions of the equation $x^{2n} + y^{2n} = 1$, for the area of this curve, and for the volumes of its revolution about x and y axes; etc., for different values of n.

35. N. W. MCLACHLAN and A. L. MEYERS, "The polar form of the ker and kei functions, with applications to eddy current heating," v. 18, 1934, p. 621–624. There are tables of $N_0(z)$, $\phi_0(z)$ (degrees), $N_1(z)$, $\phi_1(z)$ (degrees), for $z = [0.0(0.1)10.0; 5D \text{ to } 8D]$.

$$N_0(z) = (\ker^2 z + \mathrm{kei}^2 z)^{\frac{1}{2}}; \qquad \phi_0(z) = \tan^{-1}\frac{\mathrm{kei}\, z}{\ker z};$$

$$N_1(z) = (\ker'^2 z + \mathrm{kei}'^2 z)^{\frac{1}{2}}; \qquad \phi_1(z) = \tan^{-1}\frac{\mathrm{kei}'\, z}{\ker' z} + \tfrac{1}{4}\pi.$$

36. J. R. AIREY, "Toroidal functions and the complete elliptic integrals," v. 19, 1935, p. 177–188. The tabulation of toroidal functions which satisfy Laplace's equation and are suitable for conditions over the surfaces of anchor-rings, is most conveniently and accurately effected, especially when the argument is large, through the complete elliptic integrals of the first and second kind, K and E. $K(k^2) = K_1 \cdot \ln(4/k^{\frac{1}{2}}) - K_2$; $E(k^2) = E_1 \cdot \ln(4/k^{\frac{1}{2}}) + E_2$

Table I gives K_1 and K_2 with second differences, for h = [0.00000(0.00001)0.0001(.0001)0.001(.001)0.170; 10, or more, D]; T. II gives, to 10 or more D, with second differences, E_1 and E_2 for h = 0.00000(0.00001)0.0001(0.0001)0.001(0.001)0.100.

37. J. R. AIREY, "Bessel functions of nearly equal order and argument," v. 19, 1935, p. 233–235. There are tables, to 6D, of $J_\nu(\mu)$ for $\mu = 1(1)20$; $\nu = \overline{\mu - 1}(0.1)\overline{\mu + 1}$. Cf. no. **21**.

38. J. R. AIREY, "The Bessel function derivatives $\frac{\partial J_\nu(x)}{\partial x}$ and $\frac{\partial^2 J_\nu(x)}{\partial x^2}$," v. 19, 1935, p. 236–243. Tables to 4D or 5D for $x = 1(1)20$; $\nu = \overline{x - 1}(0.1)\overline{x + 1}$.

39. L. SILBERSTEIN, "On complex primes," v. 19, 1935, p. 1104. The table gives the decomposition of primes less than 1000 of the form $4n + 1$ into sums of two squares, part of the author's ms. table for $P \leqq 25{,}033$.

40. W. G. BICKLEY and J. NAYLER, "A short table of the functions $Ki_n(x)$, from $n = 1$ to $n = 16$," v. 20, 1935, p. 343–347. $Ki_n(x) = \int_0^\infty \frac{e^{-x\cosh u}du}{\cosh^n u}$, $Ki_0(x) = K_0(x)$, the Bessel function of the second kind, with imaginary argument, as defined by WATSON. The tables are for x = [0.00(0.05)0.2(0.1)2.0, 3.0; 9D]; $n = 1(1)16$.

41. L. SILBERSTEIN, "The roots of $\cos z = z$," v. 20, 1935, p. 528–531. Approximations to the real root, to 5D, and six complex roots, $z = x + iy$ for x negative, are found. The real root, to 8D, and 12 other complex roots for x positive, were found much earlier by T. H. MILLER, "On the numerical values of the roots of the equation $\cos x = x$," Edinburgh Math. So., *Proc.*, v. 9, 1891, p. 80–83. Compare *Scripta Math.*, v. 4, 1936, p. 100. See also R. COOPER and J. TODD, "The large roots of $\cos z = az + c$," v. 21, 1936, p. 249–262. Compare **no. 50.**

42. J. R. AIREY, assisted by L. J. COMRIE, "The circular and hyperbolic functions, argument $x/\sqrt{2}$," v. 20, 1935, p. 721–731. There are here two tables: the first, p. 722–726, of $\sin(x/\sqrt{2})$ and of $\cos(x/\sqrt{2})$ for x = [0.0(0.1)20.0; 12D]; the second, p. 726–731, of $\sinh(x/\sqrt{2})$ and $\cosh(x/\sqrt{2})$ for the same ranges. $\sin(x/\sqrt{2}) = \sqrt{2}[J_1(x) + J_3(x) - J_5(x) - J_7(x) + J_9(x) + \cdots]$ and $\cos(x/\sqrt{2}) = J_0(x) - 2[J_4(x) - J_8(x) + J_{12}(x) - J_{16}(x) + \cdots]$. Compare *Scripta Math.*, v. 4, 1936, p. 101–102.

43. J. R. AIREY, "The circular sine and cosine functions, argument $\log_e x$," v. 20, 1935, p. 731–738. Bessel, Neumann, Struve, and other related functions of imaginary order $i\nu$, in ascending powers of the real variable x, include the factor $x^{i\nu}$, the tabulation of which involves $\cos(\nu \ln x)$ and $\sin(\nu \ln x)$. There are the following four tables: T. I: $\sin \ln(1+\rho)$ and $\cos \ln(1+\rho)$ for ρ = [0.000(0.001)0.020; 10D]; T. II: $-\sin \ln(1-\rho)$ and $\cos \ln(1-\rho)$ for the same range; T. III: $\sin \ln x$ and $\cos \ln x$ for $x = 2, 5, 10, 100, 1000, \pi$, to 13D or 14D; T. IV: $\sin \ln x$ and $\cos \ln x$, for x = [0.1(0.1)20.0; 10D].

44. V. M. FALKNER, "A method of numerical solution of differential equations," v. 21, 1936, p. 624–640. In connection with the discussion of solutions of $Y''' - YY' = 0$, and of $Y'''' - YY'' = 0$, tables are given for Y, Y', Y'', Y''' for varying values of X.

45. R. C. COLWELL and H. C. HARDY, "The frequencies and nodal systems of circular plates," v. 24, 1937, p. 1046. On this page are given, incidentally, two tables; in one, the values of the modified Bessel functions $I_0(x)$, for x=0, 1, 5, 10, 20, 30; in the second the first ten roots of $J_n(x) = 0$ for n = [0(1)5; 3D], 34 of 60 values incorrect; cf. MTE **21**.

46. J. R. AIREY, "The radiation integrals $\int_x^\infty \frac{dx}{x^\alpha(e^x - 1)}$," v. 25, 1938, p. 273–282. These integrals "occur in a number of physical and astronomical problems, *e.g.*, when $n = 1$, the X-ray analysis of crystal structures; when $n = 2$, the radiative equilibrium of a planetary nebula and the temperatures of stars and novae; when $n = 3$, temperature of the nuclei of planetary nebulae and the exchange of energy between monatomic gases and solid surfaces." x = [0.0(0.1)10.0; 6D], $\alpha = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 2$. Compare v. 34, 1943, p. 602–606.

47. A. FLETCHER, "A table of complete elliptic integrals," v. 30, 1940, p. 516–519. "The following table was made in connection with work on Laplace coefficients in dynamical astronomy. Doubtless this is not the only case in which the modulus k is preferable to $\sin^{-1}k$ or k^2 as argument, and the functions of the complementary modulus k' are not required." $K = \pi/2M$. A table of K, E, and M, is given for k = [0.00(0.01)1.00; 10D]. See also Fletcher's "Tables of the two chief Laplace coefficients," R.A.S., *Mo. Notices*, v. 99, 1939, p. 259–265.

48. W. G. BICKLEY and J. C. P. MILLER, "Numerical differentiation near the limits of a difference table," v. 33, 1942, p. 1–14 + 4 folding plates. See RMT 93.

49. C. A. COULSON and W. E. DUNCANSON, "Some new values for the exponential integral," v. 33, 1942, p. 754–760. In T. I, $Ei(x)$ and $-Ei(-x)$ are given for x = [15(1)50; 10S]. B.A.A.S., *Mathematical Tables*, v. 1, London, 1931, and MATH. TABLES PROJECT, New York, *Tables of Sine, Cosine, and Exponential Integrals*, 2 v. 1940, give values of these functions for $0 < x < 15$. J. P. GRAM, "Undersøgelser angaaende Mængden af Primtal under en given Grænse," Danske Vidensk. Selskabs, *Skrifter*, s. 6, *Naturvid. og Math.*, v. 2, p. 268–272, gives $Ei(x)$ for x = [10(1)20; 20D], [5.0(0.2)20; 10D] with first differences. But $-Ei(-x)$ in the range $15 < x < 20$ had not been earlier computed. The only previous table of $-Ei(-x)$ for $x > 20$ is that of T. AKAHIRA, "Tables of e^{-x}/x and $\int_x^\infty e^{-u}du/u$ from $x = 20$ to $x = 50$," Inst. Phys. Chem. Research, Tokyo, *Sci. Papers, Table* no. 3, 1929; x = [20.00(0.02)50.00; 6S, 5S] with first and second differences. Coulson and Duncanson make no direct reference to the paper of Gram, from which it is clear that their last unit in $Ei(20)$ should be 6, not 7. "$Ei(x)$ is repeatedly needed in the evaluation of integrals concerned with nuclear structure, molecular structure and flow of heat."

50. A. P. HILLMAN and H. E. SALZER, "Roots of $\sin z = z$," v. 34, 1943, p. 575. In this article is a table of the first ten non-zero roots of $\sin z = z$, in the first quadrant, to 6D. Compare no. **41**.

Names

AIREY **19, 21, 22, 24, 25, 26, 27, 36, 37, 38, 42, 43, 46**
AIRY **1, 6**
AKAHIRA **49**
ALDIS **29**
BICKLEY **40, 48**
BISACRE **29**
BYERLY **8**
COLWELL **45**
COMRIE **42**
COOPER **41**
COULSON **49**
DALTON **16**
DENDY **32**
DUNCANSON **49**
ELLIS **6**
EMDE **29**
FALKNER **44**
FLETCHER **47**
GLAISHER **3, 7**
GOLDSTEIN **33**
GRAM **49**
GRAY **26**
HAMILTON **1**
HARDY **45**
HIGUCHI **34**
HILLMAN **50**
HILTON **16**
HOLLAND **8**
HUNT **11**
JAHNKE **29**
JONES **8**
KRAMP **3**
LAMB **8**
LEE **14**
LEHMER **7**
LEVERRIER **2**
LUBBOCK **2**
MCLACHLAN **35**
MARKOV **3**
MEYERS **35**
MICHELL **13**
MILLER, J. C. P. **48**
MILLER, T. H. **41**
MORTON **15**
MULHOLLAND **33**
NAGAOKA **12**
NAYLER **40**
NICHOLSON **32**
ONO **27**
OWEN **31**
PEARSON **14**
PERRY **8, 11**
RAYLEIGH **5, 9**
ROBB **17**
ROWLAND **10**
SALZER **50**
SAVIDGE **18**
SILBERSTEIN **39, 41**
SMITH **13**
SYLVESTER **7**
TALLQVIST **8**
TAYLOR **4**
TODD **41**
WAALS, VAN DER **16**
WARD **20**
WATSON **23, 40**
WEBB **24**
WRINCH, D. M. **28, 30, 32**
WRINCH, H. E. H. **28, 30, 32**

R. C. A.

121[A, C, D, Q, R].—WAR DEPARTMENT, *Technical Manual*, TM 5–236, *Surveying Tables*, July 10, 1940. Published: Washington, U. S. Government Office, 1940, 392 p. 13.8 × 21.5 cm. Supplement, TM 5–236, C1, July 29, 1942, 4 p., a list of corrections in Tables II, LIII, and LV. For sale by the Superintendent of Documents, Washington, D. C., $0.40.

Of the 56 tables in this volume we shall give only brief indications of the titles of the first 52. These are as follows: I–II, Common logarithms of numbers, and of the circular functions; III–V, Natural sines, cosines, tangents, cotangents; VI, Stadia reductions; VII, Elevations above sea level from barometric readings; VIII, Coefficients for temperature corrections, barometric; IX–X, Conversion of meters to feet and feet to meters; XI, Conversion of seconds of latitude and longitude to feet; XII, Tables of S; XIII, Conversion of decimals of a day to hours; XIV, Inclination corrections for 50 meter tape; XV, Curvature and refraction; XVI, Corrections for vertical angles; XVII, Common logarithms of m, spherical excess factor; XVIII, Azimuths of Polaris at elongation, 1940–1950; XIX, Conversion of arc into time, and time into arc; XX, Standard time signals; XXI, Mean astronomic refraction, with corrections; XXII, Parallax and altitude of the sun; XXIII–XXIV, Conversion of mean time to sidereal time, and of sidereal time to mean time; XXV, Arc-sin corrections for inverse position computations; XXVI, Conversion of degrees Centigrade to degrees Fahrenheit, and degrees Fahrenheit to degrees Centigrade; XXVII, Common logarithms of versed sines and external secants; XXVIII, Natural versed sines and external secants; XXIX, Methods of expressing gradients; XXX, Differences in elevations for given horizontal distances and gradients from 0° to 30°; XXXI–XXXII, Differences in elevations, and horizontal distances, for given slope distances and gradients from 0° to 30°; XXXIII, Reduction to the horizontal of distances paced on slopes; XXXIV, Radii and their logarithms; XXXV, Tangent offsets and middle ordinates; XXXVI, Tangents and externals for 1° curves; XXXVII, Corrections for tangents and externals; XXXVIII, Long chords and actual arcs; XXXIX, Corrections for subchord lengths; XL, Circumferences and areas of circles; XLI, Squares, cubes, square roots, cube roots; XLII, Conversion of inches and sixteenths to decimals of a foot; XLIII, Conversions of sixteenths of an inch to decimals of an inch; XLIV, Map scales in English measure and metric units; XLV, Minutes into decimals of a degree; XLVI, Useful constants and formulae; XLVII, The Greek alphabet; XLVIII, Lengths of circular arcs for unit radius; XLIX, Corrections to ΔY for magnification of scale; L, Reduction of geographic azimuth to grid azimuth; LI, Ageton table for meridian determinations; LII, Common logarithms of circular functions in hundredths of a degree.

The next four tables involve mils. Table LIII gives the logarithms, to 5D, of sine, cosine, tangent, cotangent, for every mil of the quadrant. Tables LIV–LV give natural values, to 5D, of these same functions, for the same range. Table LVI is for conversion of degrees to mils, and mils to degrees.

The supplement gives one correction in Table II, and 48 in Table LIII. Also two pages to replace two of the four pages of the table of Natural tangents and cotangents in mils.

The mils table calculated by ANDOYER to which reference has earlier been made (*MTAC*, p. 39, 84) may be noted once more, Service Géographique de l'Armée. *Tables de Logarithmes à cinq Décimales pour les nombres de* 1 *à* 12 000, *et pour les Lignes Trigonométriques dans le Système de la Division de la Circonférence en* 64 000 *parties égales* (*dixièmes du millième de l'artillerie*). Paris, Imprimerie Nationale, 1916; with differences and proportional parts.

See also the description (*MTAC*, p. 40) of punched cards for the six natural trigonometric functions, to 5D, for every tenth of a mil in the quadrant. Also RMT 126. In 1942 A. W. TUCKER published a sheet of four-place tables, for each 10 mils, of the six logarithmic and natural trigonometric functions.

122[A, D].—War Department, *Technical Manual*, TM 4–238, *Coordinate Conversion Tables* March 25, 1943. Printed at the U. S. Government Printing Office, Washington, D. C., 1942. 338 p. 13.8 × 21.5 cm. The tables in this volume were computed by the Mathematical Tables Project, New York. For sale by the Superintendent of Documents, Washington, D. C., $0.40.

This volume contains merely a single page of Introduction; Table I, coordinate conversion tables (p. 2–321); and Table II, eight-place table of sines and cosines of angles expressed in mils (p. 322–337). Table I gives the values of $A \sin x$ and $A \cos x$, where x is in mils, a mil being defined as one-sixteen hundredth of a right angle. $A = 1(1)100$ units, 50 units (and 10 mils) to a page. With left-hand pages giving $A \sin x$ and right-hand pages $A \cos x$, four pages are therefore required for each 10 mils, and hence 320 for the quadrant. The table is to 4D. If the straight line to a point 7654 yards distant make an angle of 81 mils with a line of reference a coordinate of the point, such as $A \cos x$, is found as 7629.81 (7600cos81 + 54cos81 = 7575.98 + 53.83). The result will be correct to within a unit in the second decimal place. When higher accuracy is required, Table II may be used, since this gives, to 8D, the values of sines and cosines, with first differences, for every mil in the quadrant. Identically the same second differences are given at the bottom of each of the pages 322–337 of the table, whereas it is only correct for page 332. This fact is undoubtedly due to the unscientific methods of the Government printer rather than to the personnel of the Mathematical Tables Project. *Note added in proof:* The necessary correction sheets (TM 4–238 C1, 3 p.) were issued in August, 1943.

123[A, F, P].—*Formulas in Gearing with Practical Suggestions.* Seventeenth ed., Second printing, Sept. 1943, Providence, R. I., Brown & Sharpe Mfg. Co., 15.1 × 22.9 cm. $1.50.

Since details of this excellent work have already been given (see RMT 88), it will now simply be noted that in this edition a correction has been made on p. 224. The gear ratio of the decimal equivalent .4737 is 9/19, instead of 9/10 given in previous editions.

124[A, N].—A. A. Nikitin, *Tablit͡sy Vychisleniĭa Proĭsentov* [Tables for calculating percentages], second ed., edited by P. P. Andreev, Moscow, 1939, 733 p. 21.5 × 13.9 cm. 18.90 roubles.

The prefaces to the first and second editions, by Andreev, occupy p. 3–7, while a statement of the purpose and description of the tables is on p. 8–13. The Table (p. 14–733) is for percentages 1%(1%)240%, and for numbers 100(1)999. Results are given to 1D. Each page of the Table is arranged in 30 horizontal lines (for percentages) and 10 vertical columns (for numbers), so that the whole range of 240 percentages occupies 8 pages for each ten numbers, and therefore the complete set of tables occupies 90 × 8 = 720 pages. Moreover, on each page there are two tables of proportional parts, for percentages on the left margin, and for numbers on the right margin, the latter being subdivided into three sections, for 10 horizontal lines each. There were 10 000 copies in this edition.

The volume may obviously also be viewed as another Russian table for obtaining the product of two numbers. Compare N. 16.

S. A. Joffe

125[C, D].—J. T. Peters, *Siebenstellige Logarithmentafel.* I: *Logarithmen der Zahlen, Antilogarithmen, Additions- und Subtractionslogarithmen nebst einen Anhang mit Formeln und Konstanten;* and II: *Logarithmen der trigonometrischen Funktionen für jede zehnte Sekunde des Neugrades, log sin und log tg von* 0ᵍ.0000 *bis* 3ᵍ.0000, *sowie log cos und log ctg von* 97ᵍ.0000 *bis* 100ᵍ.0000 *für jede Sekunde* (1ᶜᶜ = 0ᵍ.0001) *des Neugrades.* Berlin,

Reichsamt für Landesaufnahme, 1940, 2 v.; vii, 493, and vi, 666 p. 17.2 × 26 cm. 33.50 marks and 45.00 marks.

The second volume of mathematical tables by Peters was (in collaboration with Bauschinger) an Eight-place Table of the Logarithms of all Numbers from 1 to 200 000 (1910, revised ed., 1936); proportional parts were given throughout. The three principal eight-place tables of logarithms of numbers previously published were those of (1) JOHN NEWTON, London, 1658, for numbers up to 100 000; (2) SERVICE GÉOGRAPHIQUE DE L'ARMÉE, abridged from the *Tables du Cadastre* (see *MTAC*, p. 34, 85), Paris, 1891, up to 120 000; (3) MENDIZÁBAL TAMBORREL, Paris, 1891, up to 125 000 (see *MTAC*, p. 40). The Bauschinger-Peters volume naturally superseded all of these. This table was followed in 1922 by a work of Peters alone, namely a Ten-place Table of the Logarithms of all Numbers from 1 to 100 000. From this last volume the present Seven-Place Table has been prepared.

Except for 99 numbers the abridgement was readily effected. For 96 of these numbers the correct seven-place values were determined by consulting a twelve-place manuscript table containing the logarithms of numbers from 1 to 100 000 with an error of at most a unit in the twelfth place; this manuscript was already used in preparing the above mentioned work of 1910. The three remaining doubtful cases were recalculated and a table of the logarithms of the numbers 1 to 100 000 was completed, with no error in the seventh place as much as 0.5. The formula used by Peters for calculating the logarithm of primes, was the rapidly converging series

$$\log x = \tfrac{1}{2}\log(x+1) + \tfrac{1}{2}\log(x-1) + M\left(\frac{1}{2x^2-1} + \frac{1}{3(2x^2-1)} + \cdots\right).$$

While this table will be of value it should be borne in mind that no seven-place table has been published to supersede the work of 73 years ago by EDWARD SANG, *A New Table of Seven-Place Logarithms of all Numbers from* 20 000 *to* 200 000, London, Layton, 1871 (final printing 1915). Its preparation was based on a Sang fifteen-place manuscript of the logarithms of all numbers to 370 000, now in the Library of the Royal Society of Edinburgh.

The second section (p. 187–387) of the first volume under review contains a Seven-place Table of the Antilogarithms of numbers or seven-place numbers for the mantissas 00 000 to 99 999. As the basis of his calculations Peters used the work of HOLGER PRYTZ, *Tables d'Anti-Logarithmes*, published by the Royal Academy of Sciences of Copenhagen in 1886. It contains fifteen-place antilogarithms for mantissas 000(001)999. Peters finally developed a twelve-place antilogarithm table, with mantissas 00 000(00 001)99 999, in which the error in the seventh place was always less than 0.502. In preparing his seven-place table with an error in the seventh place less than 0.5 in every case, comparison was made with R. Shortrede's seven-place table of antilogarithms which is the second table in his *Logarithmic Tables to Seven Places of Decimals*, Edinburgh, 1844. There were other editions of this table but none of them, apparently, in the year 1854 which Peters gives.

The prefaces of Peters often contain interesting material, and the present one is no exception when he introduces matters now presented. One of the principally used logarithmic tables making possible the determination of both the logarithm of a number and the number corresponding to a given logarithm, exhibits also a table of antilogarithms in the solution of both problems. Why, it may be asked, does one then still calculate a table of antilogarithms? As principal reason for this, Peters remarks, we present the following: The differences of a table of antilogarithms are only about half as large as the differences of an ordinary table of logarithms. H. BRUNS writes about this in his *Grundlinien des wissenschaftlichen Rechnens* (Leipzig, 1903) p. 49: "For these reasons it were indeed very much more to the point to tabulate, not the logarithm but the inverse function 'Number $x = 10^x$.' In the usual seven-place arrangement for $\log x$ the first differences run from 43 to 435, and the group of three-place differences amounts to 48%. In the corresponding tabulation of the Number one would have on the contrary the differences running from 22 to 231, and the total of the three-figure differences is less than 37%, while the extent of the table has only increased from 180 to 200 pages." A further advantage of employing tables of antilogarithms lies in the fact

that one may more readily determine the number corresponding to a given logarithm with an antilogarithmic table than with a logarithmic.

The third section (p. 389–474) of the first volume is devoted to Tables of Addition and Subtraction Logarithms. For the calculation of these the fundamental tables were those of M. H. Andoyer, "Tables fondamentales pour les logarithmes d'addition et de soustraction," *Bull. Astronomique*, s. 2, v. 2, 1922, p. 5–32. They give sixteen-place values of A and S, for $D = 0.00(0.01)9.00$. (Compare RMT 116). With this as a basis Peters developed ten-place tables for A with $D = 0.0000(0.0001)6.94$; and for S with $D = 0.3000(0.0001)6.94$. The corresponding seven-place tables made from these had an error less than 0.502 in the seventh place. The final checking to assure that no error in the seventh place was as much as half a unit was carried through by G. Witt, who had made an important contribution to the Anhang of Peters and Stein in Peters' *Zehnstellige Logarithmentafel*, v. 1.

If $D = \log a - \log b$, and $\log a > \log b$, from the A table, p. 389–434, is found $\log (a + b) = \log a + A$.

In connection with the S table, p. 435–474, there are two cases. When $D > 0.3000$, $\log (a - b) = \log a - S$. If, however, $\log a - \log b < 0.3000$, set $\log a - \log b = S$ and find the D corresponding to S in the S table, then is $\log (a - b) = \log a - D$.

At the top of every page of the first of the four tables to which we have referred is $\log N$; of the corresponding pages of the second table, N; of the third table, A; and of the fourth table, S. In the right-hand columns of all of the pages are differences, d, and at the bottoms of the pages, proportional parts.

The final section of the first volume (p. 475–493) is devoted to an admirable collection of Formulas and Constants [goniometry (higher trigonometry); series; plane and spherical trigonometry; solutions of quadratic, cubic and linear simultaneous equations; theory of errors].

The second volume contains seven-place tables of the logarithms of the trigonometric functions, sine, tangent, cotangent, cosine, for each $0^g.001$. (Compare *MTAC*, p. 33–39). As basis for preparation of this table Peters used his Ten-place Table of the Logarithm of the Trigonometric functions from 0° to 90° for each 0°.001. Since $0^g.01 = 0°.009$ he took every ninth value from the ten-place table and then inserted in each interval $0^g.01$ nine new values at intervals $0^g.001$. Thus a ten-place table was made for values of log cos $0^g.000$ to log cos $50^g.000$, log tan $3^g.000$ to log tan $50^g.000$ and log sin $3^g.000$ to log sin $50^g.000$ for each thousandth of a grade, or for each tenth centesimal second. The range $0^g.000$ to $3^g.000$ receives special treatment where tables of the usual $S = \log(\sin w/w)$ and $T = \log(\tan w/w)$ are introduced for determining $\log \sin w = S + \log w$ and $\log \tan w = T + \log w$, w being in grade units. The preparation of the ten-place table of S and T called for twelve-place calculation. The still missing value for each log cot was at once written down by simply subtracting the corresponding value of log tan from ten.

The next step was to derive from this manuscript a seven-place table in which the error in the seventh decimal would be at most 0.005 of a unit. In seeking out all the doubtful values, comparison was made with the eight-place table derived from the *tables du cadastre*, namely: *Tables des Logarithmes à huit Décimales des Nombres entiers de* 1 *à* 120 000 *et des Sinus et Tangentes de dix Secondes en dix Secondes d'Arc dans le Système de la Division Centésimale du Quadrant*, Paris, 1891. In this extraordinarily correct table Peters found only two errors, the first of which he states as follows: log sin $25^g.830 = 9.596\,26245$, should be $9.596\,26245'$, so that log sin $25^g.830 = 9.596\,2624$. The second error noted, in log cot $34^g.536$, is one of those published in 1891 by Mendizábal Tamborrel, and reprinted in *MTAC*, p. 85.

This table occupies p. 136–665. Proportional parts are at the bottoms of the pages throughout. The pages 1–133 are occupied with a seven-place table of logsin w and logtan w, for the range $w = 0^g.0000$ to $w = 3^g.0000$, for each centesimal second. G. Witt was again a calculator for this volume. It does not seem as if the typographical display of either volume could have been improved.

The preface of the first volume is dated Berlin, February, 1940; and of the second, July, 1940. These volumes were received at the Copyright Office of Library of Congress on June 12, and December 3, 1940, respectively. There are film copies in the Library of Brown

University. We have yet to learn whether or not the marvellous Peters published one or more other volumes in the final year of his life.

R. C. A.

126[D].—Applied Mathematics Panel, National Defense Research Committee, *Tables of Trigonometric Functions to five significant figures and for every tenth of a mil*, AMP Report no. 24.1, September, 1943. x, 320 p. 21.2 × 27.4. Printed by the photo-offset process from manuscript. This volume is only available to members of the Panel.

This table for every tenth of a mil in the quadrant (6400 mils = 360°), gives the values for sine (5D), tangent (5D), cotangent (4D), and cosine (5D), on left-hand pages, and for secant (5D), $\frac{1}{2}$ tangent (5D), $\frac{1}{2}$ cotangent (4D), cosecant (4D) on right-hand pages. The table is based on original calculations and is the most extensive table yet published of the natural trigonometric functions with the mil as argument; compare RMT **121**, **122**. Such tables are of importance in ballistics.

The preparation of this table was started with values of trigonometric functions, taken from Peters' *Seven-Place Values of Trigonometric Functions for every Thousandth of a Degree* (see RMT **79**) at intervals of 4 mils = .225 degrees from 0° to 45°. With interpolations a five-place table was obtained for $\sin x$, $\cos x$, $\tan x$, and $\cot x$, at intervals of one mil. The values of tenths of mils were obtained by linear interpolation from the values at intervals of one mil. Among formulae used was $\sec x = \cos x(1 + \tan^2 x)$. In making a final revision the Mathematical Tables Project of New York computed $\sin x$ and $\cos x$ for $x = [0(1)200; 8D]$ where x is expressed in mils. These were then subtabulated (using linear interpolation) for tenth-mil intervals. Values of $\cot x$, $\frac{1}{2}\cot x$, and $\csc x$ were then obtained directly by division. These five functions were then rounded off to five decimal places and punched into International Business Machines cards. And so on, with numerous checks, I. B. M. equipment being used throughout.

R. C. A.

127[D].—Hans Hof, *Seven Place Full Natural Trigonometric Tables*, [v. 1] *Sine*, [v. 2] *Tangent*, [v. 3] *Cosine*, [v. 4] *Cotangent*, Ann Arbor, Mich., Box 470, University Research Assoc., 1943. Each volume contains 184 p. (including the title page). 21.1 × 27.9 cm. $4.00 per v. The v. numbers indicate merely the order of publication; v. 4 will not be published before next month.

On the back of each title page is "copyright 1943 By Hans Hof," and on the opposite page of each volume is the following: "All values have been checked carefully and found correct within approximately one point in the last or seventh place. To insure this and the many values given, this book has been lithoprinted from the original. The author and publisher will appreciate any notification of errors." In no volume is there any other text.

The three volumes contain 552 pages of tables of the natural sines, cosines, and tangents, to 7D, for every second of the quadrant. Thus each page is devoted to 15 minutes, and the table runs from the bottom to the top of each page, the volume being turned sideways for use. Under each minute are the values of the functions for 0″ followed by twelve solid groups of 35 figures for each 5 seconds. Exceptions to this statement occur in the first five or six degrees of the sine and tangent volumes where only some of the required zeros were typed, or inserted by hand, in the manuscript reproduced; there is no uniformity of procedure in this regard. For the most part, in the sine and cosine volumes, the decimal point occurs only before the first and last numbers of each column, but where zeros are inserted at random, a decimal point is usually placed before them. In the tangent volume decimal points are scattered indiscriminately over the page. There are a number of cases in top and bottom lines where the decimal points are missing. There are no rulings and the top and bottom margins of each page are less than half an inch wide. In general the

printing is clear but there is great variation between dark and light on different pages, or even on the same page. Not a few figures are blurred or indistinct, but the context usually indicates what they must be. One of several places where this is not the case (in the reviewer's copy at least) is at sin 66° 24′ 11″.

The arrangement of the sine-cosine volumes, for example, is the very inconvenient one of Gifford's table, even though this form is especially in demand in optical establishments. Throughout these two volumes, the sines always occur at the tops of the pages, 0°–89°, and the seconds, 0″–60″ in the extreme left-hand columns. Thus the sines and cosines of the same angles must be sought in different places. The curiously titled v. 1 and v. 3 will naturally be bound together.

The author has not responded to the editors' request for information as to his method of computing or compiling his tables. Do they involve any original calculations whatever? Or were they, perhaps, the result of merely rounding off values in volumes of GIFFORD (see *MTAC*, p. 9), or in the volume of PETERS (see RMT **78**, and **128**)? There is evidence against the Peters surmise, since a test sample led to so many differently rounded values.

This is the first printed seven-place table of these functions for every sexagesimal second of arc. During the last five years at least, the British Nautical Almanac Office has been in possession of a completed manuscript of such tables, with the cotangent, prepared by PETERS and COMRIE. It is most earnestly hoped that the Director of the Nautical Almanac may see his way clear to arrange for publication of this manuscript at an early date.

Hof's work was published in this country almost simultaneously with the American edition of the Peters-Comrie volume of *Eight-place Table of the Trigonometric Functions for every sexagesimal Second of the Quadrant* (see N **6** and RMT **128**). The latter publication is of proved accuracy to the last decimal place, and it is admirably arranged and printed. Unit errors in the seventh places of Hof's tables are readily shown to be extremely numerous, the whole work is very poorly printed and ill arranged, and the price is decidedly excessive.

R. C. A.

128[D].—J. T. PETERS, *Eight-Place Table of Trigonometric Functions for every sexagesimal Second of the Quadrant. Achtstellige Tafel der trigonometrischen Funktionen für jede Sexagesimalsekunde des Quadranten. . . . Published and distributed in the Public Interest by Authority of the Alien Property Custodian under License No. A-152.* Berlin, Landesaufnahme, 1939. Ann Arbor, Michigan, Edwards Brothers, 1943. xii, 901 p. 21.2 × 27.9. Photo-lithoprint reproduction. $20.00.

The original of this work, as well as the British War Office reprint, has been already reviewed in *MTAC*, p. 11–12, 65. At the latter reference, last April, we urged "most strongly" that the Alien Property Custodian arrange for the reproduction of this great work of Peters and Comrie. For, it would then be available not only for scientific workers in this country but also for our British friends, since copies of their War Office reproduction can not be procured by those not in Government employ. It is therefore with satisfaction that we greet this October publication.

The 901 pages of the table are reproduced exactly as in the original. The new title-page is indicated above; to the copyright notice on the back of the title-page has now been added, "Copyright vested in the Alien Property Custodian, 1943, pursuant to law." Instead of the original German we now have an English Preface and an English Introduction, each occupying a single page. These exact translations from the German were prepared by Mr. L. J. Comrie, although there is no reference to this fact in the volume; the wording is naturally quite different from that of the corresponding parts of the War Office edition.

The type page and side margins are just as large as in the original, but the height of the volume has been reduced by more than half an inch. In turning over the 900 pages the reviewer did not notice more than about a score of places where the figures were blurred or indistinct, and in not more than four of these cases was it impossible to guess what the

numbers must be. The green buckram binding, with inked title on the back, is stronger than that of the original. There were 1000 copies in the present edition.

Of course the publishers do not intentionally mislead when they advertise that the work was published in 1939 at $24.00, because it was priced at 60.00 Marks. Any member of the American Mathematical Society, however, and many others, could have bought this volume for $15.00, plus transportation charges. Of the four copies of this work which I know of reaching America, the cost, apart from the one sent to Library of Congress for copyright purposes, was less than $18.00 a copy. The cost of reproducing such a large work must be considerable. But might not the return to the publishers have been greater if it had been listed at $12.00? However that may be, there can be no doubt as to the importance of having this publication available at the present time.

R. C. A.

129[I, K].—R. L. ANDERSON and E. E. HOUSEMAN, *Tables of Orthogonal Polynomial Values Extended to N = 104*, Ames, Iowa, Iowa State College of Agriculture and Mechanic Arts, 1942. 14.4 × 22.2 cm. (*Research Bulletin* 297, p. 593–672, Agricultural Experiment Station, Statistical Section.)

The orthogonal polynomials considered in this *Bulletin* are

$$\xi_0' = \lambda_0\xi_0 = \lambda_0\{1\}, \qquad \xi_1' = \lambda_1\xi_1 = \lambda_1\left\{x - \frac{n+1}{2}\right\},$$

$$\xi_2' = \lambda_2\xi_2 = \lambda_2\left\{\xi_1^2 - \frac{n^2-1}{12}\right\}, \qquad \xi_3' = \lambda_3\xi_3 = \lambda_3\left\{\xi_1^3 - \frac{3n^2-7}{20}\,\xi_1\right\},$$

(1)

$$\xi_4' = \lambda_4\xi_4 = \lambda_4\left\{\xi_1^4 - \frac{3n^2-13}{14}\,\xi_1^2 + \frac{3(n^2-1)(n^2-9)}{560}\right\},$$

$$\xi_5' = \lambda_5\xi_5 = \lambda_5\left\{\xi_1^5 - \frac{5(n^2-7)}{18}\,\xi_1^3 + \frac{15n^4 - 230n^2 + 407}{1008}\,\xi_1\right\},$$

where x is a variable which takes the values 1, 2, 3, $\cdots$, n, and the λ_r, which depend on n, (except for $\lambda_0 = 1$), are so chosen that the values of the ξ_r', ($r = 1, 2, \cdots, 5$), corresponding to the n values of x are integers reduced to lowest terms. For example, taking $n = 4$, the values of ξ_3 corresponding to $x = 1, 2, 3, 4$ are $-3/10$, $9/10$, $-9/10$, and $3/10$, respectively, so that by choosing $\lambda_3 = 10/3$ the corresponding values of ξ_3' are -1, 3, -3, and 1, respectively. It is clear that $\lambda_1 = 1$ or 2 according as n is odd or even. If x takes any sequence of equispaced values $x_1, x_2, x_3, \cdots, x_n$, then ξ_1 can be defined by

$$\xi_1 = \left[x - \frac{x_1 + x_n}{2}\right] \Big/ (x_2 - x_1)$$

and the ξ_r' are then given by equations (1) in terms of this variable. It is evident that x can be assumed to take the values 1, 2, $\cdots$, n without loss of generality.

The principal use of these orthogonal polynomials is in fitting a polynomial in x by the method of least squares. Let $y_1, y_2, \cdots, y_n$ denote the values of a variable y corresponding to the values 1, 2, $\cdots$, n, respectively, of a variable x, and let these values of y be of equal precision, that is, let σ be the standard deviation of y_x for $x = 1, 2, \cdots, n$. If the polynomial

$$Y = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k, \qquad (k < n - 1) \tag{2}$$

is to be fitted to these data by the method of least squares, the a's can be evaluated by solving simultaneously the $k + 1$ normal equations

$$\sum_x x^j y_x = a_0 \sum_x x^j + a_1 \sum_x x^{j+1} + \cdots + a_k \sum_x x^{j+k}, \qquad (j = 0, 1, 2, \cdots, k) \tag{3}$$

where Σ_x denotes summation over $x = 1, 2, \cdots, n$. The sums of powers up to the $2k^{\text{th}}$ of

the natural numbers are required, and the solution of these equations can be very laborious, especially when $k \geqslant 3$ and n is large. However, if Y is expressed as

$$Y = A_0\xi_0' + A_1\xi_1' + \cdots + A_k\xi_k' \tag{4}$$

where ξ_0', ξ_1', ξ_2', $\cdots$, ξ_k' are polynomials of degree 0, 1, 2, $\cdots$, k in x, respectively, and possessing the orthogonal property

$$\begin{cases} \sum_x \xi_r'\xi_s' = 0 & \text{for} \quad r \neq s \\ \sum_x (\xi_r')^2 \neq 0, \end{cases} \tag{5}$$

then the normal equations for the A's, namely,

$$\sum_x \xi_r' \cdot y_x = A_0 \sum_x \xi_r'\xi_1' + A_1 \sum_x \xi_r'\xi_2' + \cdots + A_k \sum_x \xi_r'\xi_k', \qquad (r = 0, 1, 2, \cdots, k), \tag{6}$$

reduce to

$$\sum_x \xi_r' \cdot y_x = A_r \sum_x (\xi_r')^2, \qquad (r = 0, 1, 2, \cdots, k), \tag{7}$$

and the determination of the A's is greatly simplified, especially when the values of the ξ''s are known for $x = 1, 2, \cdots, n$. The polynomials ξ_0', ξ_1', $\cdots$, ξ_5' defined in (1) are known to possess the orthogonal property (5), and the *Bulletin* under review makes available more extensive tables of their values, and of the values of $\Sigma_x (\xi_r')^2$, than have been available heretofore

The first fifteen pages of this *Bulletin* are devoted to methods of calculation and use, and to references cited. The tables begin on p. 610, and for $n = 3(1)104$ give the exact values of ξ_1' to ξ_5' (or, to ξ_n' for $n \leq 5$) corresponding to $x = \dfrac{n+2}{2}(1)n$, or $x = \dfrac{n+1}{2}(1)n$, according as n is even or odd; when $n < 9$, the full set of values for $x = 1(1)n$ is given. This abridged tabulation is made possible by the fact that for $x = \dfrac{n+1}{2} \pm m$ the values of ξ_2 are equal, and also of ξ_4; similarly ξ_1', ξ_3', and ξ_5' assume values which are of opposite sign. The value of $\Sigma_x (\xi_r')^2$ and of λ_r is given for each value of n and r below the corresponding column of values of ξ_r'.

The portion of these tables for $n = 3(1)52$ is a reproduction, with some variation of format, of Table XXIII in R. A. FISHER and F. YATES, *Statistical Tables for Biological, Agricultural and Medical Research*, Edinburgh, 1938, and contains the erratum noted in MTE 9, p. 86, namely, the sum of $(\xi_3')^2$ for $n = 39$ should read "4,496,388" and not "496,388" as tabulated. This error has been corrected in the second edition (1943) of the source table.

P. L. CHEBYSHEV[1] considered the determination of such polynomials for the general case where the y's are of varying weight and x takes any set of n distinct values. For the case where x takes any n equispaced values Chebyshev listed[2] polynomials φ_0, φ_1, φ_2, φ_3, φ_4, φ_5, satisfying (5) and gave[2] a recurrence formula relating φ_r to φ_{r-1}, φ_{r-2}, and φ_1 for $r \geqslant 2$. In consequence polynomials satisfying (5) are sometimes referred to as Chebyshev polynomials, but this terminology is reserved, by specialists in orthogonal polynomials, for another set of orthogonal polynomials which he investigated also.[3]

Much later the subject was treated independently by F. ESSCHER,[4] by C. JORDAN,[5] and by R. A. FISHER.[6] FISHER gave explicit expressions[6,7] for polynomials ξ_0, ξ_1, ξ_2, $\cdots$, ξ_5 when the y's are of equal weight and x takes any equispaced values, which become the expressions in braces in (1) when x takes the values 1, 2, $\cdots$, n. The coefficients of x_r in ξ_r are always unity. Miss F. E. ALLAN[8] pointed out that FISHER's ξ_r are related to CHEBYSHEV's φ_r by

$$\xi_r = \frac{r!}{(2r)!}\varphi_r, \qquad (r = 0, 1, 2, \cdots). \tag{8}$$

Making this substitution in CHEBYSHEV's recurrence formula[3] and advancing the subscript to $r + 1$ it follows that

$$\xi_{r+1} = \xi_1 \xi_r - \frac{r^2(n^2 - r^2)}{4(4r^2 - 1)} \xi_{r-1}. \tag{9}$$

Miss ALLAN,[8] using equation (11) below, has provided a simple proof of this recurrence formula. It has been used in the preparation of this *Bulletin* to find the values of ξ_{r+1} for $x = 1, 2, \cdots, n$ from the corresponding values of ξ_r and ξ_{r-1}. The value of λ_{r+1} was then determined and the ξ'_{r+1} values obtained from $\xi'_{r+1} = \lambda_{r+1} \cdot \xi_{r+1}$.

Starting from CHEBYSHEV's expression[9] of φ_r as proportional to the r^{th} difference of the product

$$x(x-1)(x-2)\cdots(x-r+1)(x-n)(x-n-1)\cdots(x-n-r+1),$$

Miss ALLAN derived two general formulae (p. 316 and p. 319) for ξ_r, the latter being given in the third and subsequent editions of FISHER's book,[7] and she tabulated the polynomials ξ_r up to ξ_{10}. The corresponding general formulae have been derived by A. C. AITKEN[10] for the polynomials

$$T_r = \frac{(2r)!}{(r!)^2} \xi_r = \varphi_r/(r!)^2, \qquad (r = 0, 1, 2, \cdots), \tag{10}$$

the values of which are always integers for $x = 1, 2, 3, \cdots, n$. AITKEN also gave procedures for building up the polynomial values from terminal values and differences. These procedures have been used by the authors of this *Bulletin* to check the values of ξ_r obtained by means of the recurrence relation (9).

CHEBYSHEV[9] obtained a formula for $\Sigma_x \varphi_r^2$, which in terms of the ξ_r becomes

$$\sum_x \xi_r^2 = \frac{(r!)^4}{(2r)!(2r+1)!}(n+r)(n+r-1)\cdots(n+1)n(n-1)\cdots(n-r) \tag{11}$$

$$= \frac{(r!)^4}{(2r)!(2r+1)!} n(n^2-1^2)(n^2-2^2)\cdots(n^2-r^2), \qquad (r = 0, 1, 2, \cdots).$$

This formula is derived directly by Miss ALLAN,[8] and the corresponding formula for the T_r is derived by AITKEN.[10] A modification of this formula was used by the authors of this *Bulletin* to compute the values of $\Sigma_x (\xi_r')^2$ given in the tables.

The reviewer has been surprised to notice that the authors of this *Bulletin* do not even mention CHEBYSHEV, whose contributions to the subject are clearly indicated in the papers by AITKEN and ALLAN which they cite.

C. EISENHART

[1] P. L. CHEBYSHEV, "Sur les fractions continues," *Jn. d. Math.*, s. 2, v. 3, 1858, p. 289–323. *Oeuvres*, St. Petersburg, v. 1, 1899, p. 201–230.

[2] CHEBYSHEV, "Sur l'interpolation par la méthode des moindres carrés," Akad. Nauk, Leningrad, *Mémoires*, s. 7, v. 1, 1859, p. 1–24. *Oeuvres*, v. 1, p. 471–498.

[3] G. SZEGÖ, *Orthogonal Polynomials*, Am. Math. So., *Coll. Publ.*, v. 22, New York, 1939.

[4] F. ESSCHER, "Ueber die Sterblichkeit in Schweden, 1866–1914," Lund, Observatoriet, *Meddel.*, s. 2, v. 23, 1920, p. 10–21.

[5] CH. JORDAN, "Sur une série de polynomes dont chaque somme partielle répresente la meilleure approximation d'un degré donné suivant la méthode des moindres carrés," London Math. So., *Proc.*, s. 2, v. 20, 1920, p. 297–325.

[6] R. A. FISHER, "Studies in Crop Variation.—I. An examination of the yield of dressed grain from Broadbalk," *Jn. Agric. Sci.*, v. 11, 1921, p. 107–135.

[7] R. A. FISHER, *Statistical Methods for Research Workers*, Edinburgh, third ed., 1930.

[8] F. E. ALLAN, "The general form of the orthogonal polynomials for simple series, with proofs of their simple properties," R. So. Edinburgh, *Proc.*, v. 50, 1930, p. 310–320.

[9] CHEBYSHEV, "Sur l'interpolation," Akad. Nauk, *Zapiski*, v. 4, 1864. *Oeuvres*, v. 1, p. 539–560.

[10] A. C. AITKEN, "On the graduation of data by the orthogonal polynomials of least squares," R. So. Edinburgh, *Proc.*, v. 53, 1933, p. 54–78.

130[I, K].—TRUMAN LEE KELLEY (1884–), *The Kelley Statistical Tables*, New York, Macmillan, 1938, v, 136 p. 21.5 × 27.8 cm. $4.50.

This book contains six different statistical tables, of which the first is the largest and most important.

Table I. The area (p) under the normal curve is given by

$$p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt \tag{1}$$

and its ordinate (z) by

$$z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \tag{2}$$

Equation (1) may be inverted to yield x as a function of p. This inverse function $x(p)$ is called by statisticians the normal deviate corresponding to the probability level p. If $x(p)$ is substituted in equation (2) the ordinate z becomes a function, $z(p)$, of the area p to which it is the boundary ordinate.

The table gives, in parallel columns with argument p, the functions $x(p)$, $z(p)$ and the algebraical expressions $\sqrt{pq}$, $\sqrt{1-p^2}$ and $\sqrt{1-q^2}$, where $q = 1 - p$. All five functions are tabulated for p = [0·5000(0·0001)0·9999; 8D]. The complementary argument q (which statisticians call the complementary probability level) is given in the right-hand column. No differences are given. The table covers 100 well-printed pages of convenient size; the layout is good, although we should have preferred the eight decimals divided into the more conventional groups of five and three, rather than four and four.

Most tables of the normal curve—e.g. (a), (c), (h) and (l)—give p and z as functions of x. The inverted form of normal deviates $x(p)$ is fairly common in statistical usage; see, for example, (b), (g) and (j), but the ordinate z as a function of the probability p has little or no application, although it is given in (j). KONDO and ELDERTON'S 10-decimal normal deviates $x(p)$, which are also given in (j) for p = 0·500(0·001)0·999, formed the starting-point of the present tables. Both (j) and KELLEY'S Tables go beyond the present-day requirements of users; four-decimal accuracy in x and z is all that is wanted for most applications. The fine interval of 0·0001 in p is certainly a convenience for some problems such as the calculation of probits (d)[1] in toxicology, but for statistical routine work of the nature of tests of significance a three-decimal (if not a two-decimal) argument is all that is required; the one-page tables given in (b) and (g) are handier to use.

The functions $\sqrt{pq}$, $\sqrt{1-p^2}$ and $\sqrt{1-q^2}$ are entirely independent of the columns x and z. The author states that he has included them in the same table simply because the requisite arguments were available. The functions $\sqrt{1-p^2}$ and $\sqrt{1-q^2}$ are important in correlation work,[2] particularly when partial correlations have to be computed from correlations of the next lower order from formulae such as

$$r_{12\cdot3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}.$$

The function $\sqrt{pq}$ is useful in work with the binomial distribution, in particular for the calculation of standard deviation $\sqrt{pq}/\sqrt{N}$ of a percentage p based on a sample of N observations. When $\sqrt{Npq}$ is required, it is often more convenient to multiply q by the integer Np (the observed number having a certain characteristic) and use a table of square roots.

Table II is a small 2-page table of the incomplete Γ-function, known to statisticians as the probability integral P of the statistic χ^2 (i) (k), and given by

$$P = \frac{2(\frac{1}{2}n)^{\frac{1}{2}n}}{(\frac{1}{2}n-1)!} \int_{\chi\sqrt{n}-\frac{1}{2}}^{\infty} x^{n-1} e^{-\frac{1}{2}nx^2} dx.$$

Four decimals are given for P, while the parameter n (called the degrees of freedom) takes the values 1(1)10, 12, 15, 19, 24 and 30. The argument $\chi/\sqrt{n}$ has the range 0·0(0·1)4·1; for n greater than about 12 or 15 the interval of 0·1 is too large.

Tables III, IV and V. Since no differences are given in Tables I and II, auxiliary tables of Lagrangian coefficients are provided for interpolation on a calculating machine. The range of argument is 0 to 1; further details are given below.

Table	called	points	interval	decimals
III	Cubic	4	0·001	10 (exact)
IV	Quintic	6	0·01	10
V	Septic	8	0·1	11 (exact)

Unfortunately these tables are bound in with the main tables, so that it is awkward to use them for interpolation in Tables I and II. They would have been more convenient under a separate cover. Errors are discussed in MTE 24. The accuracy of the various interpolation formulae is fully discussed and demonstrated over the whole range of the table. Inverse interpolation becomes necessary, as the author wishes to cover cases when $p(x)$ or $z(x)$ are required; the processes described are very cumbersome, because the simple methods of inverse interpolation (e), which make use of printed differences, are not applicable here.

Table VI gives square roots of all integers from 1 to 1000, to 7 decimals up to 99, and thereafter to 6; in other words, there are always 8 figures. The layout, in which consecutive values are in lines rather than in columns, is not convenient.

In conclusion it may be said that the value of these tables lies in the accurate record of normal deviates at a very fine interval. This should be useful in fundamental work on statistical distributions derived from the normal law. The working statistician, on the other hand, wants a larger number of statistical functions to fewer decimals, rather than a few functions to many decimals.

Some Related Tables

(a) W. F. Sheppard, *The Probability Integral* (B.A.A.S., *Mathematical Tables*, v. 7), Cambridge, University Press, 1939.
(b) R. A. Fisher and F. Yates, *Statistical Tables*, second ed., London, Oliver and Boyd, 1943, Table I.
(c) Fisher and Yates, *idem*, Table II.
(d) Fisher and Yates, *idem*, Table IX.
(e) L. J. Comrie, *Interpolation and Allied Tables*. Reprinted from the *Nautical Almanac* for 1937. London, H. M. Stationery Office, 1936, p. 934–940.
(f) J. R. Miner, *Tables of* $\sqrt{1-r^2}$ *and* $1-r^2$. Baltimore, Johns Hopkins, 1922.
(g) K. Pearson, *Tables for Statisticians and Biometricians*, part I, third ed., London, Biometrika, 1930, Table I.
(h) K. Pearson, *idem*, Table II.
(i) K. Pearson, *idem*, Table XII.
(j) K. Pearson, *Tables for Statisticians and Biometricians*, part II. London, Biometrika, 1931, Table II.
(k) K. Pearson, *Tables of the Incomplete* Γ*-function*. London, Biometrika, 1922.
(l) Mathematical Tables Project, New York, *Tables of the Probability Integral*, v. 2, 1942.

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H. O. Hartley

[1] The quantity $x + 5$ is equivalent to a table of probits as used in biometric work.

[2] Miner's table (f) of $\sqrt{1-r^2}$ gives 6 decimals for $r = 0{\cdot}0000(0{\cdot}0001)1{\cdot}0000$. Its layout is less convenient than that of the present table.

131[K, L].—Edwin Raymond Smith (1879–), "Zeros of the Hermitian [*sic*] polynomials," *Amer. Math. Mo.*, v. 43, 1936, p. 354–358.

In Hermite's original papers, "Sur un nouveau développement en série des fonctions," Institut de France, Acad. Sci., *Comptes Rendus*, v. 58, 1864, p. 93–100, 266–273, his polynomials were defined by

$$U_n(x) = e^{x^2}\frac{d^n e^{-x^2}}{dx^n}$$

which satisfies the equation $U_n'' - 2xU_n' + 2nU_n = 0$.

Mr. Smith's Hermitean polynomials are defined, like those in Jahnke and Emde, compare RMT **113**, and in W. Hahn's "Bericht"[1] by

$$H_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n e^{-\frac{1}{2}x^2}}{dx^n},$$

and are of the form

$$H_n(x) = x^n - \frac{n(n-1)}{2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} x^{n-4} - \cdots,$$

$$U_n(x) = (-\sqrt{2})^n H_n(\sqrt{2}x).$$

Mr. Smith states that he had not been able to find in print any list of numerical values of the roots of Hermitean polynomials, and that apparently none had been computed except for special cases. Hence he computed the zeros for $n = [3(1)27; 6D]$.

The functions of the parabolic cylinder, $u_n(x)$ are connected with the Hermitean polynomials by the equation

$$u_n(x) = e^{-\frac{1}{4}x^2} H_n(x)$$

which satisfies the differential equation

$$u_n''(x) + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)u_n(x) = 0.$$

Since $H_n(x)$ and $u_n(x)$ vanish for the same values of x, methods due to Sturm are applied to this equation for obtaining first approximations to the zeros for small values of n. He remarks that while this procedure is perfectly general it is easier for relatively large values of n to obtain the approximations by extrapolation from the zeros of polynomials corresponding to $n - 1$, $n - 2$, $\cdots$, by well-known difference methods.

He then notes that better values may be obtained from these first estimates by Newton's method of successive approximations. For comparatively small values of n, however, a direct application of this method is not feasible; in this case he indicates an adaptation of the method suitable for machine calculation.

R. C. A.

[1] W. Hahn, "Bericht über die Nullstellen der Laguerreschen und der Hermiteschen Polynome," Deutsche Mathem.-Ver., *Jahresb.*, v. 44, 1934, p. 215–236. It is here pointed out, following Hermite, that the $H_n(x)$ first occurred in 1836 in a paper of Sturm, and that in 1839 they are incidentally mentioned in a paper by Chebyshev.

J. B. Russell (*Jn. Math. Phys.*, M.I.T., v. 12, 1933, p. 291–297) has defined a Hermite function by the relation $\phi_n(x) = 2^n e^{-\frac{1}{4}x^2} H_n(x)$ and given a table of $\phi_n(x)$ for $n = 0(1)11$, and $x = [0.00(0.04)1.00(0.10)4.00(0.20)7.00, 7.50, 8.00$; to at least 5S].

132[K, L].—E. R. Smith and Archie Higdon, "Zeros of the Legendre polynomials," Iowa State College, *Jn. Sci.*, v. 12, 1938, p. 263–274. Compare RMT **92**, and **118**.

In this table are given the non-negative zeros of $P_n(x)$, to 6D, from $n = 2$ to $n = 40$ inclusive. The zeros for $n = 2$ to $n = 7$ were abridged from the zeros to 15D in J. W. L. Glaisher's article, B.A.A.S., *Report*, 1879, p. 53. These zeros may have been found seven years earlier. In any case they antedate the values of these same zeros to 16 D, given in Heine, *Kugelfunktionen*, v. 2, Berlin, 1881, p. 15–16, and are in essence equivalent to those given by Gauss in 1816.

In 1934 Mr. Higdon computed the zeros of the polynomials from $n = 8$ to $n = 25$ of the present article. He used a method based on some theorems by Sturm, referred to in RMT **131**. Mr. Smith's computations were for zeros of polynomials from $n = 26$ to $n = 40$. For this purpose he developed two formulae (p. 263–271) which may be used to compute approximate values for the zeros. One of these formulae is intended primarily for the values near zero and the other for values near unity, either of the formulae being suitable for computing the middle values. The method by which the formulae were obtained is an adaptation of the one which was used in a paper by F. Zernike ("Eine asymptotische Entwickelung für die grösste Nullstelle der Hermiteschen Polynome," Akad. v. Wetens., *Proc.*, v. 34, 1931, p. 673–680).

Unit errors in Mr. Higdon's zero values, from $n = 8$ to $n = 16$, found upon comparing them with those of A. N. LOWAN, *et al.*, RMT 92, are as follows: $n = 8$, *for* .525533, *read* .525532; $n = 11$, *for* .519095, *read* .519096; $n = 12$, *for* .367832, *read* .367831; $n = 14$, *for* .827202, *read* .827201; $n = 16$, *for* .095012, *read* .095013, and *for* .281605, *read* .281604.

R. C. A.

133[L].—A. N. LOWAN and MILTON ABRAMOWITZ, "Table[s] of the integrals $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$," *Jn. Math. Phys.*, M.I.T., v. 22, 1943, p. 2–12. 17 × 25.2 cm.

Accompanying these tables is not a single reference to the literature of the subject. Integrals of Bessel functions have occurred for a long time in physical investigations and until recently adequate tables have been unavailable. In KOPPE's work on the propagation of waves along a weighted string[1] and in ORR's work on radiation from an alternating current,[2] integrals of Bessel functions of integral order occurred. Estimates of the value for large values of the order and argument were given by POLLACZEK.[3] Integrals of the Bessel functions of the second kind occurred in the work of HAVELOCK[4] on ship waves and wave resistance. The need of tables was met to some extent by the 7-place tables of WATSON[5] which covered the range $x = .02(.02)1$ and included values at the first 16 zeros of the integrand $J_0(t)$ or $Y_0(t)$. Four-place tables for the same range were given by GLAZENAP.[6] Graphs of the first integral, divided by x, were given in the papers of DEBYE[7] and FISCHER.[8] This first integral has occurred recently in some work on the intensities of electron diffraction rings by BLACKMAN[9] who gives an equivalent definite integral

$$R(A) = \int_0^\infty \sin^2 [A(1 + w^2)^{\frac{1}{2}}]dw/(1 + w^2) = \int_0^A J_0(2t)dt$$

and a plot of $R(A)$ which indicates that a maximum value 7.2 occurs when A is about 1.25. This function $R(A)$ is a particular case of a function of 4 variables studied by BUCHHOLZ[10] in his work on the spreading of alternating currents in the earth.

The tables under review are given for the range $x = [0(.01)10; 10D]$, and are a by-product of the computation of Bessel functions of complex arguments. Power series are used for the first integral while the computation of the second integral is effected with the aid of the first integral multiplied by a logarithm and two power series the first of which is evaluated by iterated Taylor expansions, by a method similar to that described in the Introduction of *Tables of Sine, Cosine and Exponential Integrals*, v. 1, prepared by the MATHEMATICAL TABLES PROJECT.

H. B.

[1] M. KOPPE, *Die Ausbreitung einer Erschütterung an der Wellenmaschine durch einen neuen Grenzfall der Besselschen Functionen*, Progr. Andreas Realgymn., Berlin, 1899; see also *Forts. Math.*, v. 30, p. 420–421.

[2] W. Mc. F. ORR, "Note on the radiation from an alternating circular electric current," *Phil. Mag.*, s. 6, v. 7, 1904, p. 336–341.

[3] F. POLLACZEK, "Über die Fortpflanzung mechanischer Vorgänge in einem linearen Gitter," *Annalen d. Physik*, s. 5, v. 2, 1929, p. 991–1011.

[4] T. H. HAVELOCK, "Studies in wave resistance: the effect of parallel middle body," R. So. London, *Proc.*, v. 108 A, 1925, p. 77–92; "Ship waves: their variation with certain systematic changes of form," R. So. London, *Proc.*, v. 136 A, 1932, p. 465–471.

[5] G. N. WATSON, *Theory of Bessel Functions*, Cambridge, Univ. Press, 1922, p. 752.

[6] S. P. GLAZENAP, *Matematicheskie i Astronomicheskie Tablitsy*, Leningrad, Acad. Sci., 1932, p. 97.

[7] P. DEBYE, "Zerstreuung von Röntgenstrahlen," *Annalen d. Physik*, s. 4, v. 46, 1915, p. 809–823.

[8] F. A. FISCHER, "Über die akustische Strahlungsleistung von Strahlengruppen insbesondere der Kreis- und Kugelgruppen," *Elek. Nach. Tech.*, v. 9, 1932, p. 147–155.

[9] M. BLACKMAN, "On the intensities of electron diffraction rings," R. So. London, *Proc.*, v. 173A, 1939, p. 68–82.

[10] H. BUCHHOLZ, "Die Wechselstromausbreitung im Erdreich unterhalb einer einseitig offenen und unendlich langen vertikalen Leiterschleife im Luftraum," *Archiv f. Elektrotechnik*, v. 30, 1936, p. 1–33.

134[L].—ARNOLD NOAH LOWAN (1898–), GERTRUDE BLANCH, and MILTON ABRAMOWITZ (1915–), "Tables of $Ji_0(x) = \int_x^\infty J(t)dt/t$ and related functions," *Jn. Math. Phys.*, M.I.T., v. 22, 1943, p. 51–57. 17 × 25.2 cm.

Since this paper contains practically no reference to the literature of the subject, it may be well to give indications in this regard, so as to articulate what is new in the tables under review. Integrals of the type $\int J_n(x)x^n dx$ were studied in a general way by Lommel[1] but the first use of the integral tabulated in this paper may have been in 1889 when the brothers LODGE[2] obtained the integral

$$\int_0^A [1 - J_0(x)]dx/x$$

in some physical investigations. The integral $Ji_0(x)$ (see Edinb. Math. So., *Proc.*, s. 2, v. 3, p. 276) was transformed into a definite integral by LERCH[3] in 1896. The result involves the gamma function of a complex variable and a parameter w whose real part lies between 0 and $\frac{1}{2}$. The notation adopted here is essentially that of B. VAN DER POL[4] who showed that the equation

$$\partial J_n(x)/\partial n = \int_0^\infty Ji_0(x - t)J_n'(t)dt \quad n \geqslant 1$$

could be used to calculate the derivative of the Bessel function with respect to n. The suffix n was added by P. HUMBERT[5] when the Bessel-integral function was defined for an arbitrary value of n by the equation

$$Ji_n(x) = -\int_x^\infty J_n(t)dt/t.$$

Humbert gave a generating function for the functions of integral order

$$lie^{\frac{1}{2}x(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n Ji_n(x)$$

and obtained a recurrence relation

$$(n - 1)Ji_{n-1}(x) - (n + 1)Ji_{n+1}(x) = (2n/x)J_n(x).$$

He also gave many series and definite integrals for the Bessel-integral functions. In 1938 HOGG[6] obtained an expansion for $\int_x^\infty J_0(t)dt/t^{2r+1}$ which is derived again by the present authors for the special case $r = 0$.

The asymptotic expansions which are given in this paper, and in the adjoining one of V. G. SMITH,[7] appear to be new. The coefficients are calculated here to a large number of significant figures so that the series can be used for accurate computations. Smith's expansion is convenient for $0 < x < 25$ while that of this paper is thought to be better for $x > 25$.

In Table I the function $F(x) = Ji_0(x) + \ln(\frac{1}{2}x)$ is tabulated with $\delta^2 F(x)$, $\delta^4 F(x)$ and $\delta^6 F(x)$, for $x = [0(.1)3; 10D]$. The function $Ji_0(x)$ is tabulated with central differences δ^2 and δ^4 in Table II, for $x = [3(.1)10; 10D]$, and without differences for $x = 10(1)22$, the values for $x = 0(.1)3$ being given in Table I. Table III contains reduced derivatives of $F(x)$; the function $A_n(x)$ which is tabulated for $n = [0(1)13; 12D]$, $x = 10(1)21$, is defined by the equations

$$A_n(x) = F^{(n)}(x)/n!, \qquad A_0(x) = F(x).$$

These new tables will be much welcomed.

H. B.

[1] E. LOMMEL, "Über eine mit den Bessel'schen Funktionen verwandte Funktion," *Math. Ann.*, v. 9, 1876, p. 425–444.

[2] O. J. LODGE and A. LODGE, "The rotation of the plane of polarization of light by the discharge of a Leyden jar," *Phil. Mag.*, s. 5, v. 27, 1889, p. 339–346.

[3] M. LERCH [Speculations on some questions in integral equations] (Bohemian), Česká Akad., *Rozpravy, Class II*, v. 5, 1896, No. 23, 16 p.; *Forts. Math.*, v. 27, 1896, p. 233–235.

[4] B. VAN DER POL, "On the operational solution of linear differential equations and an investigation of the properties of these solutions," *Phil. Mag.*, s. 7, v. 8, 1929, p. 861–898.

[5] P. HUMBERT, "Sur les fonctions de Bessel-intégrales," Inst. d. France Acad. d. Sci., *Comptes Rendus*, v. 195, 1932, p. 854–855; "Bessel-integral functions," Edin. Math. So., *Proc.*, s. 2, v. 3, 1933, p. 276–285.

[6] A. H. A. HOGG, "Equilibrium of a thin plate, symmetrically loaded, resting on an elastic foundation of infinite depth," *Phil. Mag.*, s. 7, v. 25, 1938, p. 576–582.

[7] V. G. SMITH, "An asymptotic expansion of $Ji_0(x) = \int_x^\infty J_0(t)dt/t$," *Jn. Math. Phys.*, M.I.T., v. 22, 1943, p. 58–59.

135[L].—A. N. LOWAN, and WILLIAM HORENSTEIN, "On the function $H(m, a, x) = \exp(-ix)F(m + 1 - ia, 2m + 2; ix)$." *Jn. Math. Physics*, M.I.T., v. 21, 1942, p. 264–283.

This paper contains tables of confluent hypergeometric functions for certain complex arguments. When normalized in a certain way, the confluent hypergeometric functions are also called Whittaker functions; see WHITTAKER and WATSON, *Modern Analysis*, fourth ed., Cambridge, 1927, and Amer. ed., New York, 1943, chapter 16.

In a systematic study of second order ordinary differential equations, the hypergeometric functions play an essential rôle. If the differential equation possesses less than three regular singular points, the solution can be obtained in elementary functions; with three regular singularities the solution, while no longer expressible in terms of elementary functions, can be solved by hypergeometric functions. When two of the regular singularities coalesce, an irregular singularity is obtained, and the resulting equation has confluent hypergeometric functions as solutions.

In applications such differential equations have arisen in a number of connections. H. A. WEBB and J. R. AIREY, in *Phil. Mag.*, s. 6, v. 36, 1918, p. 129–141, suggested a number of applications of the confluent hypergeometric function in the theory of elasticity, and in problems of flexure and vibration, and published tables and graphs of the function for certain real arguments. See *MTAC*, "Mathematical tables in *Phil. Mag.*," no. **24**. For further tables published by AIREY see "Mathematical Tables in Reports of the B.A.A.S.," nos. **41**, **43**, *MTAC*, p. 73. Later, R. GRAN OLSSON discussed the confluent hypergeometric function in connection with the buckling of elastic plates of variable thickness, and the whirling of shafts; and included further tables. (*Zeit. angew. Math. Mech.*, v. 16, 1936, p. 347–348; *Ingenieur-Archiv*, v. 8, 1937, p. 81–98, 99–103, 270–275, 373–380.) A collection of curves is found in JAHNKE and EMDE, *Tables of Functions*, New York, 1943, p. 275–282.

All these applications and tables involved only real arguments and real values of the parameters. In the quantum theory of atomic collisions, i.e., in the solution of the Schrödinger wave equation for a Coulomb field, the confluent hypergeometric function occurs with imaginary arguments and certain complex values of the parameters. (See A. SOMMERFELD, *Ann. Physik*, s. 5, v. 11, 1931, p. 257–330, especially p. 272–273; N. T. MOTT and H. S. W. MASSEY, *Theory of Atomic Collisions*, Oxford, 1933, p. 36.) The present tables have been prepared with special reference to these quantum mechanical applications.

The confluent hypergeometric function y may be defined by the series

$$F(\alpha, \gamma; u) = 1 + \frac{\alpha}{\gamma}\frac{u}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{u^2}{2!} + \cdots + \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{\gamma(\gamma+1)\cdots(\gamma+n)}\frac{u^{n+1}}{(n+1)!} + \cdots.$$

The corresponding differential equation is

$$uy'' + (\gamma - u)y' - \alpha y = 0.$$

The tables give seven-place values of $H(m, a, x)$, as defined in the title of the paper, for $x = 0(1)10$ and for $a = 0(1)10$, $m = 0(1)3$.

P. W. KETCHUM

136[L].—A. N. LOWAN, and H. E. SALZER, "Table of coefficients in numerical integration formulae," *Jn. Math. Phys.*, M.I.T., v. 22, 1943, p. 49–50. 17 × 25.2 cm.

The Bernoulli polynomial $B_n^{(m)}(x)$ of order m and degree n may be defined by means of the generating function

$$t^m e^{xt}(e^t - 1)^{-m} = \sum_{n=0}^{\infty} t^n B_n^{(m)}(x)/n!$$

Its properties are given very fully by NÖRLUND,[1] MILNE-THOMSON,[2] and DAVIS.[3] When $m = n$ there is an integral representation

$$B_n^{(n)}(x) = \int_x^{x+1} (t-1)(t-2)\cdots(t-n)dt.$$

When $x = 0$ the value of the polynomial is denoted by the symbol $B_n^{(n)}$ and the numbers $B_n^{(n)}$ are connected by the recurrence relation

$$\sum_{s=0}^{n} \binom{n}{s} B_s^{(s)}/s! = (-1)^n B_n^{(n)}/n!$$

Values of these numbers for $n = 1(1)5$ were given by NÖRLUND on p. 192 of his paper, and on p. 459 of his book there are tables for $n = 1(1)12$, values being given also for $B_n^{(n)}/n!$.

On p. 209–210 of his book DAVIS gives expressions for $B_n^{(m)}(0)$ for $n = 0(1)12$ and the values of $B_n^{(n)}$ may be deduced by putting $m = n$. In the present tables both $B_n^{(n)}(0)$ and $B_n^{(n)}(1)$ are given for $n = 1(1)20$, the values for the higher suffixes being given to twenty or more significant figures. The numbers occur as coefficients in Laplace's formulae of numerical integration. The latter set occurs when forward differences are used as in the Gregory formula of interpolation, and the former set when backward differences are used. The new values provide for much greater accuracy in numerical integration.

H. B.

[1] N. E. NÖRLUND, "Mémoire sur les polynomes de Bernoulli," *Acta Math.*, v. 43, 1922, p. 121–196; *Vorlesungen über Differenzenrechnung*, Berlin, Springer, 1924.
[2] L. M. MILNE-THOMSON, *The Calculus of Finite Differences*, London, Macmillan, 1933, p. 181–184, 191–192.
[3] H. T. DAVIS, *Tables of the Higher Mathematical Functions*, Bloomington, Ind., The Principia Press, v. 2, 1935, p. 208–210.

137[L].—J. A. STRATTON, P. M. MORSE, L. J. CHU and R. A. HUTNER, *Elliptic Cylinder and Spheroidal Wave Functions*, New York, Wiley; London, Chapman & Hall, 1941, xii, 127 p. 21 × 27.2 cm. [P. 1–51 are an exact reprint of an article by Chu and Stratton in *Jn. Math. Physics*, M.I.T., v. 20, Aug. 1941, p. 259–309. To this were added a title-page and foreword (p. i–xii) and "Errata" sheet by Morse, and tables, p. 53–125, by Stratton, Morse, Chu, Hutner, J. B. Fisk, and J. R. Pelham; the introduction to the tables, by Hutner, occupies p. 53–76. All the extra pages are almost wholly reproductions of typed material.] $1.00.

The linear differential equation of the second order which, in the normal form, has no term involving the first derivative, unity as the coefficient of the second derivative and a periodic function $P(t)$ as the coefficient of the dependent variable y, is usually called Mathieu's equation, or the equation of the elliptic cylinder, when $P(t)$ is of the form $a + 16q\cos 2t$. The special case in which $a = 1$ was used by Laplace in 1777 to illustrate his method of solving differential equations approximately.[1] He first expanded the solution in powers of q, neglecting terms of order higher than q^2, and obtained two linearly independent solutions involving trigonometrical functions of t some of which were multiplied by t and t^2. Not content with this form of solution he adopted another method [1] not unlike the method of

the variation of arbitrary constants, and obtained a solution in which trigonometrical functions of odd multiples of t up to the fifth were multiplied by exponential factors $e^{\pm 4qt}$. Laplace thus introduced the idea of a characteristic index which for his case and for the order of approximation adopted was $\pm 4q$. At this time he was interested in the question of the stability of the solar system and his thoughts on this question may have led to the search for this second form of his solution. Laplace pointed out, moreover, that his method was applicable also to the more general equations with periodic coefficients which occur in astronomical investigations. The equation in which $P(t)$ has the form of a cosine series of even multiples of t was studied in detail a hundred years later by G. W. Hill and is commonly called Hill's equation because he introduced a novel method of treatment based on the use of infinite determinants, a method which was made rigorous by Henri Poincaré and von Koch.[2]

These linear differential equations with periodic coefficients became important also in acoustics and were used by Rayleigh in his paper on the maintenance of vibrations by forces of double frequency.[3] Many other writers like A. Stephenson and C. V. Raman have used these equations in the study of the stability of a dynamical system or of some peculiarities of the oscillations.

Another very extensive field in which equations of this type occur was opened up when in 1837 G. Lamé [4] focussed attention on the problem of the separation of variables in the study of the equation of heat conduction, and in the case of thermal equilibrium studied the simple solutions of the partial differential equation having the form of a product of three functions, each of a single variable, now called a Lamé product. In 1868 this idea was used by E. Mathieu [5] in the study of the vibrations of an elliptic membrane and it became important to find when the differential equation had a periodic solution. Such solutions are now known as Mathieu functions and the associated harmonic functions which occur in the solution of potential problems are called the functions of the elliptic cylinder. In 1873 Mathieu used elliptic coordinates also [5] for the thermal problem in which there is conduction of heat and showed that the separation of variables leads to a more general equation which includes the former equation as a special case. This equation is the one on which the theory of spheroidal wave functions is based. It was studied in some detail by C. Niven [6] in 1880 while Heine said [7] in 1881 that he had been unable to complete the attempt to use in the general case the method which proved successful in the treatment of the equation of the elliptic cylinder in his first volume.

Heine succeeded, indeed, in classifying the periodic solutions of Mathieu's equation into four classes, the trigonometrical functions in the expansion used being either all sines or all cosines and the multiples of t either all even or all odd. This classification has been adopted by Whittaker and others as it suggests a convenient notation which, however, is not used by the present writers.

In his early work on Lamé functions Heine used expansions in series of associated Legendre functions and in his later work on the functions of the elliptic cylinder he used also expansions in series of Bessel functions. This plan was adopted by Niven for the study of the spheroidal wave functions and to a slight extent by M. Brillouin [8] in 1904. Its advantages were shown by E. G. C. Poole in 1923 and were particularly emphasized by E. A. Hylleraas [9] in his lectures in 1937 at the Institute of Henri Poincaré. Not only is the approximation more rapid than when power series are used (as in the work of Maclaurin and the associated work of Brillouin) but the recurrence relations by which the coefficients are calculated are essentially the same. This fact seems to depend on the circumstance that the desired functions are solutions of a homogeneous integral equation discovered by M. Abraham in 1899 and in a more general form by E. T. Whittaker,[10] who also obtained a corresponding result for Lamé functions.

The associated Legendre functions and Bessel functions used in the expansions are limiting forms of the spheroidal wave functions and the use of these expansions is analogous to the method of W. Ritz for the direct solution of a problem in the Calculus of Variations. It is perhaps on this account, as Hylleraas points out, that the method of computation is so effective.

This method is essentially that used in the work under review except that the related functions of Gegenbauer are used in place of the associated Legendre functions, the notation being different from that used by Nielsen.[11] A new notation applicable also to the elliptic cylinder functions is adopted for the spheroidal wave functions and special values of the functions are listed. It is necessary to consider both prolate spheroidal functions and oblate spheroidal functions. The functions are also classified according to the type of Gegenbauer function or Bessel function used in the expansion. Eight U-functions $U_n(a, c; z)$ are defined by means of series of different types of Bessel functions and eight V-functions $V_n(a, c; z)$ are defined by means of different types of series of Gegenbauer functions. Questions of convergence are discussed, expansions are obtained in the form of power series and relations are obtained among the U-functions, among the V-functions and between the U-functions and V-functions. The corresponding solutions of the wave-equation are denoted by an S-notation and an R-notation. Different kinds of functions are distinguished which in turn are classified as even or odd. Combinations are formed also to correspond to the Hankel functions and serve for the solution of radiation problems in electromagnetic theory. Functions are also described as angular or radial, the angular functions being periodic in t where $z = \cos t$. The separation constants are calculated with the aid of continued fractions by a method due to Lindstedt, which was used successfully by Hough in 1897 in an application of harmonic analysis to the dynamical theory of the tides. This method was recommended by Poole in 1923 and was used successfully by Ince and Goldstein in their work on the tabulation of Mathieu functions from 1927 on.

Another change of notation is effected by the introduction of a normalization factor, and the coefficients D_n^l which are tabulated to 5D in the first 5 tables for $l = 0(1)4$ are the coefficients in the expansion in a series of Bessel functions $J_n(cz)$ of a function $Je_l(c, z)$ which is a multiple of the U-function of the Mathieu type. Seven coefficients are tabulated and values are given also for the separation constant b_l. The next 4 tables give for $l = 1(1)4$ the values of the coefficients in the expansion in a series of Bessel functions $J_n(cz)$ of a function $Jo_l(c, z)$ which is a multiple of the $U_3(c, z)$ function of the Mathieu type. Seven coefficients and the separation constants are given. In the list of errata a misprint is noted in the equation (310). The values of the separation constants supplement those of Goldstein and Ince as they are for small values of c and small interval while c is used as variable instead of a multiple of c^2.

The next table gives the separation constant A, as in the equation on p. 62, for the prolate spheroidal functions, to 5D, for $c = 0(.1).6(.2)5$. The meaning of the suffixes is not given; one of them is presumably m, the other l. The values $m = 0, 1, 3$ are chosen, l being zero in the last case and ranging from 0 to 3 in the first case and from 0 to 2 in the second. This meaning is more or less confirmed by the next table of the coefficients d in the expansions of the S- and R-functions for the prolate spheroid. The range of c is the same as before but m also has the value 2 which is missing in the table of the A's. About 16 coefficients are given for each case. The remaining tables refer to the oblate spheroidal wave functions. The table of B's on p. 107 refers to the B in the equations on p. 69, and it is to be noted that the value $m = 2$ is now given in the table but only with $l = 1$, whereas in the table of the coefficients f the value $l = 0$ is also considered.

The tables are evidently intended to be used chiefly for the solution of electromagnetic problems. In the problems of wave mechanics connected with the hydrogen molecule ion the internal equation is the same as the equation considered in the present paper but the external equation is a little different. The short table calculated by Hylleraas is consequently not comparable with these tables. The only comparison that can be made is with the few numerical results obtained by Maclaurin[12] and M. Brillouin[8] in their work on electrical oscillations and with the new results obtained recently by Page and Adams.[13] It can, then, be said that these tables form a distinct addition to our knowledge of spheroidal wave functions.

H. B.

[1] P. S. DE LAPLACE, "Mémoire sur l'intégration des équations différentielles par approximation," *Mém. Acad. roy. sci.*, Paris, 1780, or *Oeuvres*, v. 9, p. 357–380, especially p. 359–362, 372–374.

[2] *A Semi-Centennial History of the American Mathematical Society 1888–1938*, New York, 1938, p. 119–120.

HELGE VON KOCH, "Sur une application des déterminants infinis à la théorie des équations differentielles linéaires," *Acta Math.*, v. 15, 1891, p. 53–63; "Sur les déterminants infinis et les équations differentielles linéaires," *Acta Math.*, v. 16, 1892–3, p. 217–295.

[3] RAYLEIGH, "On maintained vibrations," *Phil. Mag.*, s. 4, v. 15, 1883, p. 229–235, or *Scientific Papers*, v. 2, 1900, p. 188–193; "On the maintenance of vibrations by forces of double frequency, and on the propagation of waves through a medium endowed with a periodic structure," *Phil. Mag.*, s. 4, v. 24, 1887, p. 145–159, or *Scientific Papers*, v. 3, 1902, p. 1–14.

[4] G. LAMÉ, "Mémoire sur les surfaces isothermes dans les corps solides homogènes en équilibre de température," *Jn. de Math.*, s. 1, v. 2, 1837, p. 147–188; "Mémoire sur l'équilibre des températures, dans les corps solides homogènes . . .," *Jn. de Math.*, s. 1, v. 4, 1839, p. 126–163, 351–385.

[5] E. L. MATHIEU, "Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique," *Jn. de Math.*, s. 2, v. 13, 1868, p. 137–203; *Cours de Physique Mathématique*, Paris, 1873.

[6] C. NIVEN, "On the conduction of heat in ellipsoids of revolution," R. So. London, *Trans.*, v. 171, 1880, p. 117–151.

[7] HEINE, *Handbuch der Kugelfunktionen*, second ed., Berlin, v. 2, 1881.

[8] M. BRILLOUIN, *Propagation de l'Électricité*, Paris, Hermann, 1904, Ch. 6, p. 376, 383–4, etc.

[9] E. A. HYLLERAAS, "Équation d'ondes d'un électron dans le champ de forces de deux noyaux atomiques, problème de l'ion moléculaire d'hydrogène," Inst. Henri Poincaré, *Annales*, v. 7, 1937, p. 121–153.

[10] E. T. WHITTAKER, "On the functions associated with the elliptic cylinder in harmonic analysis," Intern. Congress Math., Cambridge, 1912, v. 1, p. 366–371. "On Lamé's differential equation and ellipsoidal harmonics," London Math. So., *Proc.*, s. 2, v. 14, 1915, p. 260–268 or *Modern Analysis*, p. 564; "On an integral equation whose solutions are the functions of Lamé," R. So. Edinburgh, *Proc.*, v. 35, 1915, p. 70–77. See also J. H. PRIESTLEY, "On some solutions of the wave equation," London Math. So., *Proc.*, s. 2, v. 20, 1922, p. 37–50. J. L. SHARMA, *Jn. de Math.*, s. 9, v. 16, 1937, p. 199–203, 355–360.

[11] N. NIELSEN, *Théorie des Fonctions Metasphériques*, Paris, 1911.

[12] R. C. MACLAURIN, "On the solutions of the equation $(\nabla^2 + k)\psi = 0$ in elliptic coordinates and their physical applications," Cambridge Phil. So., *Trans.*, v. 17, 1898, p. 41–108. For numerical results see p. 92–95, 98, 106, 108.

[13] L. PAGE and N. I. ADAMS, JR., "The electrical oscillations of a prolate spheroid, part I," *Phys. Rev.*, s. 2, v. 53, 1938, p. 819–831.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **121** and **122** (WAR DEPT.), **123** (BROWN & SHARPE), **125** (SERVICE GÉOGRAPHIQUE), **132** (HIGDON), and in **118** (HOLLAND, JONES and LAMB; BYERLY); see also in the second article of this issue, nos. **8** and **45**. For errors made by MEISSEL and WATSON see the first article in this issue.

21. J. BOURGET, {*Tables of the first nine roots of* $J_s(x_n) = 0$, $s = [0(1)5; 3D]$}, Paris, École Normale Sup., *Annales*, v. 3, 1866, p. 82–87.

On comparing the 54 entries of these tables to 3D, with the tables to 5S or 6S for $s = [0(1)10(5)20(10)50 \cdots 1000]$, of J. R. AIREY, B.A.A.S., *Report*, 1922, p. 271 (see *MTAC*, p. 72), it was found that 27 of the Bourget entries were erroneous. Four of the worst errors were as follows: $J_3(x_9)$, *for* 32.050, *read* 32.065; $J_4(x_9)$, *for* 33.512, *read* 33.537; $J_6(x_1)$, *for* 8.780, *read* 8.772; $J_5(x_9)$, *for* 34.983 *read* 34.989. All 27 of these errors are faithfully reproduced in each of the five editions of JAHNKE and EMDE, *Table of Functions*; see RMT **113**. So also for Table V (p. 302) in GRAY and MATHEWS, *Treatise on Bessel Functions*, second ed., London, 1922; for Table XXXIII (p. 82) in J. B. DALE, *Five-Figure Tables of Mathematical Functions*, London, 1903; and for Table V (p. 286) in W. E. BYERLY, *An Elementary Treatise on Fourier's Series* . . . , Boston, 1895. These 27 errors (and one more added) are reproduced with equal faithfulness in RAYLEIGH, *The Theory of Sound*, v. 1, London, 1877, p. 274 (also in the German ed., v. 1, Brunswick, 1879, p. 364, and in

various other English prints such as v. 1, 1929, p. 330.) Compare J. R. AIREY, *Phil. Mag.*, s. 6, v. 32, 1916, p. 10. F. BOWMAN, *Bessel Functions* (1938) also copied Bourget errors.

R. C. A.

22. [L. J. COMRIE], *Seven-Figure Trigonometrical Tables for every Second of Time*, London, 1939. Compare *MTAC*, p. 43.

In addition to the errors already included in the Errata Slip are the following: P. 74, d for tan $2^h\ 05^m$, *for* 995–1000, *read* 995–1001; d for cot $3^h\ 54^m$, *for* 1000–995, *read* 1001–995.

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23. P. R. E. JAHNKE and F. EMDE, *Tables of Functions*, first ed., 1909. Compare RMT **113**.

P. 54, $F(35°, 30°)$, *for* 0.6220, *read* 0.6200. P. 137, table expressing $J_0(x\sqrt{i})$ in the form $a + ib$, there are 10 errors of sign in the imaginary parts from $x = 5.1$ to $x = 6.0$ inclusive (F. F. P. BISACRE, *Phil. Mag.*, s. 6, 1923, v. 45, p. 1035.)

24. T. L. KELLEY, *The Kelley Statistical Tables*, 1938. Compare RMT **130**.

The Tables III, IV and V (of 4-point, 6-point and 8-point Lagrangian coefficients) have been compared with tables of these coefficients that I made about 12 years ago on a Burroughs machine. In the 2000 values of Table III there is only one error:

Page 121, $p = 0.155$, col. C_0, *for* .90033 79375, *read* .90033 69375. As the values are exact, the question of rounding-off errors does not arise. In Table IV there are no errors, when we take into account the fact that in 15 cases the rounding off has been (quite properly) changed by a unit in order that the sum of the coefficients shall be 1. The 72 values in Table V are all exact and correct.

L. J. C.

25. MATHEMATICAL TABLES PROJECT, New York, I. *Tables of the Exponential Function e^x*, 1939; II. *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*, 1940; compare RMT **89**; III. *Tables of Sine, Cosine and Exponential Integrals*, v. 1, 1940; IV. *Table of Natural Logarithms*, v. 4, 1941; V. *Tables of Probability Functions*, v. 1, 1941; compare RMT **91** and MTE **16**; VI. *Table of Arctan x*, 1942; compare RMT **90**.

I. P. 202, heading, *for* "descending," *read* "ascending"; p. 282, heading, *for* o^x, *read* e^{-x}; p. 335, argument, *read* .5486.

II. P. 358, argument, *read* 1.7850.

III. P. xv, footnote, *for* "result in error by six units," *read* "result in error by two units"; also, *for* $(1/24)p(p-1)(p-2)(p-3)$, *read* $(1/24)p(p^2-1)(p-2)$.

IV. P. 440, argument, *read* 9.3890.

V. P. 131, argument, *read* 0.6496.

VI. P. 36, arguments, *read* 3.456, and p. 71, *read* 6.905.

A. N. LOWAN

26. C. W. MERRIFIELD, "The sums of the series of the reciprocals of the prime numbers and of their powers," R. So. London, *Proc.*, v. 33, 1881, p. 4–10.

Mr. K. W. Miller, of Chicago, has called my attention to the following errata (p. 10):

n	*for*	*read*
3^1	17096	17476
5	50164	50174
11	69104	69105
13	26973	26983

There is also a last figure error in Merrifield's preceding table (p. 9) of $\log S_n$ for $n = 13$ where 1 should be 2. All these errata are in Gram's reproduction of these tables [K. Danske Videns. Selskabs, *Skrifter*, s. 6, *Naturd. og Math.*, v. 2, p. 269]. Their discovery is the result of collation with Davis' excellent tables of these sums to 24 decimals in his *Tables of the Higher Mathematical Functions*, Bloomington, Ind., v. 2, 1935, p. 249–250. The verification mentioned by Glaisher (*l.c.*) has been applied to the tables of Davis and Glaisher by the writer without uncovering any error.

D. H. L.

[1] The first error was discovered by J. W. L. Glaisher, *Quart. Jn. Math.*, v. 25, 1891, p. 373, and listed in the writer's *Guide to Tables in the Theory of Numbers* (without due credit).

27. J. T. PETERS, *Sechsstellige Tafel der trigonometrischen Funktionen*, . . . , Berlin, 1929.

In acknowledging the list of errors in this work which I had sent to Peters (see MTE **14**), he wrote on 23 May 1939 that a new proof had been read for the second edition (which appeared later in 1939), and that this had confirmed his suspicion that Herr Schoch, who is mentioned in the Introduction as one of the readers, had not been sufficiently painstaking with his share (except in the case of the last three figures of each function), and that many errors remained. I felt that this warning should be passed on to anyone who may be using the first edition.

L. J. C.

28. HERMANN BRANDENBURG, *Sechsstellige trigonometrische Tafel*, Leipzig, Lorentz, 1932. xxiv, 304 p. 19.9 × 28.3 cm.

This table gives 6-figure natural values of the four principal trigonometric functions at interval 10″, with an auxiliary table of cotangents up to 3° at interval 1″. Like the similar 7-figure tables by the same author (Leipzig, Lorentz, 1923 and 1931), it is based on the *Opus Palatinum* of Rheticus (1596). It was compared soon after publication with Andoyer's *Nouvelles Tables Trigonométriques Fondamentales* (Paris, Hermann, 1915 and 1916). The early cotangents were compared with the values computed by the undersigned for Peters' *Achtstellige Tafel der trigonometrischen Funktionen für jede Sexagesimalsekunde des Quadranten* (Berlin, Reichsamt für Landesaufnahme, 1939; Amer. ed., Ann Arbor, Mich., 1943). These comparisons and other examinations led to the discovery of the errors listed below, only the first of which is serious. This work is, therefore, of a very high standard of accuracy, whereas the first edition of the 7-figure tables (1923) was full of errors. Brandenburg, however, by offering rewards for any errors discovered, and in other ways, laboured incessantly to remove this blemish. The second edition of the 7-figure table (1931) was almost free from error, and this volume has fewer errors than the corresponding volume by Peters of all six functions to six figures at the same interval (see MTE **14, 27**).

page	angle, etc.	*for*	*read*
4	cot 0°16′24″	208,6171	209,6171
95	Argument col. for P.P. 1340	3 (line 2)	2
150	tan 20°09′10″	Delete asterisk	
193	Arguments 27°18′40″ and 50″	To be interchanged	
225	P.P. for 7/10 of 168	117,7	117,6
302	Beispiel, line 3	mau	man
300	ρ°	170 32	170 33
300	ρ'	219 2	219 9
300	ρ''	152	197

The errors in ρ°, ρ' and ρ'' are due to multiplying $1/\pi$ (which is correct) by 180, 10,800 and 648,000, and writing too many significant figures. There are also end-figure errors in $(\frac{1}{2}\pi)^n$, especially for the higher values of n. The value of e on the same page is, of course, correct to 19 decimals only, since the largest neglected term begins in the 20th decimal.

L. J. C.

UNPUBLISHED MATHEMATICAL TABLES

We have also referred to unpublished mathematical tables of (a) SILBERSTEIN, in the second article of this issue; of (b) PETERS, in RMT **125**, and in N **13**; and of (c) PETERS and COMRIE in RMT **127**.

15[A].—SOLOMON ACHILLOVICH JOFFE (1868–) *Table of the first 100 factorials, and the method of its computation.* Ms., prepared in January 1921, in possession of the author; in 1921 photographic copies were placed in the libraries of the Edinburgh Mathematical Society, the University of Edinburgh, and the Royal Society of Edinburgh, and in 1943, in the Library of Brown University. Compare UMT **10**.

A table of the first 50 factorials is contained in a paper by FRANK ROBBINS, "Factorials and allied products with their logarithms," R. So. Edinb., *Trans.*, v. 52, part 1, 1917, p. 167–174. This factorial table was extended by the writer shortly after it was brought to his attention. Beginning with $n = 51$, the multiplications (and additions for verification) were made on the Electric Ensign Machine. Each value $n!$ was obtained from $(n-1)!$ through multiplication by n, the multiplicand being broken up into 9-figure groups. The resulting product was verified by the congruence $n! \equiv 0 \pmod{10^9 - 1}$. E.g. for 68! 2480 + 035542 + 436830 + 599600 + 990418 + 569171 + 581047 + 399201 + 355367 + 672371 + 710738 + 018221 + 445712 + 183296 = 5999994.

Each quinary value $n!$, for $n = 5k$, was computed a second time from $(n-5)!$ through multiplication by $n^{(5)}$, the latter values having been obtained by successive addition from the table of $0^{(5)}$, $5^{(5)}$, $10^{(5)}$, $15^{(5)}$, $20^{(5)}$ and $25^{(5)}$. In multiplying $(n-5)!$ by $n^{(5)}$, the latter was taken as the multiplicand and $(n-5)!$ was separated into 6-figure groups for multipliers. The final value 100! was also verified by the congruence $100! \equiv -1 \pmod{101}$.

S. A. JOFFE

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16[A, E, L].—*Miscellaneous Tables.* Mss. prepared by, and in possession of, the MATHEMATICAL TABLES PROJECT, 50 Church St., New York City. None of the items listed below is intended for publication.

1. Square roots of the integers up to 100 to about 18S.

2. The functions $x^n/n!$ and $x^n/n(n!)$ for $x = [0.00(0.01)2;\ 15D]$, and $n = 1, 2 \cdots p$, where p depends on x and is such that the functions in question are of the order of magnitude of 10^{-13}.

3. Values of $x^n/n!$ for $x = [1(1)10;\ 10S]$, and $n = 1(1)40$.

4. The first fourteen derivatives of the function $(1/\sqrt{2\pi})e^{-\frac{1}{2}x^2}$, for $x = [0(0.1)8.4;\ 20D]$.

5. The first fourteen derivatives of the function $(2/\sqrt{\pi})e^{-x^2}$ for $x = [0.0(0.1)8.0;\ 20D]$.

6. The first ten derivatives of the sine and cosine integrals for $x = [10(1)100;\ 12D]$.

7. The first eighteen derivatives with respect to r of the functions $J_0(z)$ and $J_1(z)$, where $z = re^{i\theta}$, for $\theta = 0°(5°)90°$ and $r = 0.0(0.1)10$. The derivatives are given to a number of places (varying from 8 to 15) adequate for the subsequent subtabulation of the functions to about 12D.

8. The first eighteen derivatives of the functions $x^{-\nu}J_\nu(x)$ and $x^{-\nu}I_\nu(x)$ for $x = 0.0(0.1)10$ and $\nu = \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}$. These derivatives are given to a number of places (varying from 8 to 15) adequate for the subsequent subtabulation of the functions to about 12D.

9. The first fifteen derivatives of $J_\nu(x)$ and $I_\nu(x)$ for $x = 10.0(0.1)25$. These derivatives are given to a number of places (varying from 8 to 15) adequate for the subsequent subtabulation of the functions to about 12D or 12S.

10. The first eighteen derivatives with respect to r of the functions

$$S_0(z) = Y_0(z) - (2/\pi)J_0(z)(\ln z + \gamma),$$
$$S_1(z) = Y_1(z) - (2/\pi)J_1(z)(\ln z + \gamma) + 2/\pi z,$$

where $z = re^{i\theta}$, for $\theta = 0°(5°)90°$ and $r = 0.0(0.1)10$. These derivatives are given to a number of places (varying from 8 to 15) adequate for subsequent subtabulation of the functions to 12D.

11. Modulus and argument of $J_0(z)$ and $J_1(z)$, where $z = re^{i\theta}$, for $\theta = 0°(5°)90°$ and $r = [0.0(0.1)10; 10D \text{ or } 10S]$.

A. N. LOWAN

17[B].—H. T. DAVIS, *Tables of powers.* Mss. in possession of the author. Compare Q 5.

I. Tables of reciprocals, values of $1/n$, for $n = [100(1)1000; 17D]$, together with differences. The following work giving results to 7D is well known: W. H. OAKES, *Table of the Reciprocals of Numbers from* 1 *to* 100,000 . . ., London, 1865; there is also the more recent work, *Barlow's Tables* . . . , fourth ed. by L. J. COMRIE, London, 1941 (RMT 82) which gives the reciprocals of all integers up to 12500, to 7S, with differences; "Reciprocals of primes less than 10,000," in units of the tenth and eleventh decimal, are given with differences in B.A.A.S., *Mathematical Tables*, v. 5, *Factor Table*, London, 1935, p. 284–291. II. Tables of reciprocals of squares and cubes, values of $1/n^2$ and $1/n^3$, for $n = [100(1)1000; 16S–18S]$. There are tables for $n = [50(1)200; 10D–13D]$ in K. PEARSON, *Tables for Statisticians and Biometricians*, part 2, London, 1931, p. 244–245. III. Tables of reciprocal fourth and fifth powers, values of $1/n^4$ and $1/n^5$, for $n = [100(1)1000; 18S]$. The values of $1/n^p$, to 32D, for $n = 2(1)100$, $p = 1(1)16$, are given in J. T. PETERS, *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922, *Anhang*, by Peters and Stein, p. 36–57. IV. Tables of the functions $x^m(1 - x^2)^n$ for $x = [0.00(0.01)1.00; 14D]$; $m = 0$, 1, and 2; $n = 1/2$, 3/2, 5/2, 7/2, and 9/2. These tables were used in computing the values of the integrals of the Legendre associated functions. In his *Tables of* $(1 - r^2)^{\frac{1}{2}}$ *and* $1 - r^2$ *for use in Partial Correlation and Trigonometry* (Baltimore, Md., 1922), J. R. MINER gives the values of $(1 - r^2)^{\frac{1}{2}}$ for

$$r = [0.0001(0.0001).9999; \text{ mostly to 6D}].$$

H. T. D. and R. C. A.

18[L].—*Tables of the Associated Legendre Functions and their First Derivatives*, Mss. prepared by, and in possession of, the MATHEMATICAL TABLES PROJECT, 50 Church St., New York City.

These are six-place tables of the functions $P_n^m(x)$ and $Q_n^m(x)$ and their first derivatives for real and purely imaginary arguments, integral and half-integral values of n, and integral values of m. Details of the tabulation are as follows:

Tables of $P_n^m(\cos\theta)$ and $dP_n^m(\cos\theta)/d\theta$ for $\theta = 0°(1°)90°$; $n = 0(1)10$; $m = 0(1)4$; $m \leq n$.

Tables of $P_n^m(x)$, $dP_n^m(x)/dx$, $Q_n^m(x)$, $dQ_n^m(x)/dx$, for $x = 1.0(0.1)10$, $n = 0(1)10$; $m = 0(1)4$; $m \leq n$.

Tables of $i^{-n}P_n^m(ix)$, $i^{-n}dP_n^m(ix)/dx$, $Q_n^m(ix)$, and $dQ_n^m(ix)/dx$, for $x = 1.0(0.1)10$; $n = 0(1)10$; 10; $m = 0(1)4$; $m \leq n$.

Tables of $P_{n+\frac{1}{2}}^m(x)$, $dP_{n+\frac{1}{2}}^m(x)/dx$, $Q_{n+\frac{1}{2}}^m(x)$ and $dQ_{n+\frac{1}{2}}^m(x)/dx$, for $x = 1.0(0.1)10$; $n = 0(1)4$; $m = 0(1)4$ (even where m is greater than n).

Comparison of the contents of these manuscripts with the tables referred to in H. T. Davis's review of Mursi's table of $P_n^m(x)$ (RMT 118) reveals the following facts: (a) Tallqvist's tables of $P_n^m(\cos\theta)$ cover a greater range in n and give more decimals than ours. (b) The function $dP_n^m(\cos\theta)/d\theta$ is not given by Tallqvist; (c) Mursi's tables of $P_n^m(x)$ for x less than unity, are completely complementary to our tables for real x. (d) There is some slight overlapping between our values of $dP_n^m(\cos\theta)/d\theta$ and those in the Tallqvist manuscript.

A. N. LOWAN

19[L].—*Table of the Bessel Functions $K_0(x)$ and $K_1(x)$ for Small Arguments.* Ms. prepared by, and in possession of the MATHEMATICAL TABLES PROJECT, 50 Church St., New York City.

The B. A. A. S. published volume of tables of *Bessel Functions* contains a table of $K_0(x)$ and $K_1(x)$ for $x = 0.00(0.01)5$. Eight significant figures are given and second differences are tabulated for $x > 0.3$. For $x < 0.3$ the auxiliary functions $E_0(x)$, $E_1(x)$, $F_0(x)$ and $F_1(x)$ are also tabulated for purpose of interpolation; the last mentioned functions being defined from the formulae

$$K_0(x) = E_0(x) + F_0(x) \log x, \qquad K_1(x) = E_1(x)/x + F_1(x) \log x.$$

Because of the frequent use of these functions in various branches of research, the Project has subtabulated the functions above mentioned in the range from 0 to 1. The ranges and intervals for the various functions are as follows: For $K_0(x)$ and $K_1(x)$, $x = 0.0000(0.0001)$-$0.0300(0.001)1.000$. For $E_0(x)$, $E_1(x)$, $F_0(x)$ and $F_1(x)$, $x = 0.000(0.001)0.030$. All entries are given to seven significant figures with first and second advancing differences.

A. N. LOWAN

20.[L].—*Bessel Functions of the Second Kind for Complex Arguments.* Mss. prepared by, and in possession of, the MATHEMATICAL TABLES PROJECT, 50 Church St., New York City. Film copy at Brown University.

These are ten-place tables of Bessel functions of the second kind, $Y_0(z)$ and $Y_1(z)$, for complex arguments $z = re^{i\theta}$, where $r = 0.00(0.01)10$, $\theta = 0°(5°)90°$.

Bessel functions of orders 0 and 1 of the first and second kinds are encountered in the general solution of boundary value problems arising in the theory of potential, heat conduction, and wave motion, when the domain is a cylinder or a cylindrical annulus. They occur in particular in the problem of propagation of electromagnetic waves, the theory of the skin effect for poorly conducting wires, the oscillatory motion of a sphere in a viscous medium, the vibration of a heavy chain in a resisting medium, the determination of the frictional and gliding coefficients of a fluid in rotary vibrational motion and in many other boundary value problems. They have further theoretical importance in connection with the problem of conformal mapping and the evaluation of contour integrals involving Bessel functions.

No tables of Bessel functions of the second kind for complex arguments are to be found in the literature except for arguments along the 45° ray, which are known as ker and kei functions. These functions (or functions closely related to them) have been tabulated by H. G. SAVIDGE, *Phil. Mag.*, s. 6, v. 19, 1910 for $x = [1(1)30; 4S]$; by F. TÖLKE, *Besselsche und Hankelsche Zylinderfunktionen nullter bis dritter Ordnung vom Argument $r\sqrt{i}$*, Stuttgart, 1936, for $r = [0.00(0.01)21.00; 4S]$; by H. B. DWIGHT, *Tables of Integrals and Other Mathematical Data*, 1934, for $x = [0.0(0.1)10; 7D–9D]$; by N. W. MCLACHLAN and A. L. MEYERS, *Phil. Mag.*, s. 7, v. 18, 1934, in polar form, for $z = [0.0(0.1)10; 5D$ to $8D]$; by K. HAYASHI, *Fünfstellige Funktionentafeln*, Berlin, 1930, p. 111–118. Shorter tables are given by N. W. MCLACHLAN, *Bessel Functions for Engineers*, Oxford, 1934; by E. JAHNKE and F. EMDE, *Tables of Functions with Formulae and Curves*, New York, 1943; and by B.A.A.S., *Report*, 1916.

A. N. LOWAN

MECHANICAL AIDS TO COMPUTATION

8[Z].—S. L. BROWN and L. L. WHEELER, "Use of the mechanical multiharmonigraph for graphing types of functions and for solution of pairs of non-linear simultaneous equations," *Review of Scientific Instruments*, v. 13, 1942, p. 493–495.

This note indicates the application of the Brown harmonic analyzer (described in *MTAC*, p. 128–129) to the drawing of "complex Lissajous figures" and their application. Such figures are curves whose parametric equations express x and y as trigonometric polynomials in θ, such as

$$x = \cos 13\theta + \cos 15\theta, \qquad y = -\sin 13\theta + \sin 15\theta,$$

which is clearly a rose of 28 petals. Nine drawings, some of them quite complicated, are reproduced in this article.

If $f(x, y)$ and $g(x, y)$ are polynomials, the machine can be used to solve the system

$$f(x, y) = h, \qquad g(x, y) = k,$$

for x and y by setting

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{A}$$

and thereby transforming f and g into a pair of functions of r and θ, which, for a fixed (trial) value of r, become trigonometric polynomials. If the corresponding complex Lissajous figure passes through the point (h, k) for a certain value of θ, this value, and the value selected for r, determine x and y by (A). If not, a new trial value of r is interpolated.

D. H. L.

9[Z].—S. L. BROWN and L. L. WHEELER, "The use of a mechanical synthesizer to solve trigonometric and certain types of transcendental equations, and for the double summations involved in Patterson contours," *Jn. Applied Phys.*, v. 14, 1943, p. 30–36.

The methods and results of applying the Brown analyzer (described in MTAC, p. 128–129) to the problem of solving such equations as

$$x + \sin x = 3$$
$$\log(2x^3 + 5x^2 + 10x + 15) = 5/x$$
$$3\tan^2 x + 2\cos^2 x + \sin x + \csc x = 5$$

are discussed in detail in this paper. In each case a trigonometric polynomial is obtained by some device and the machine used to graph a curve from which the desired roots are read off. In the case of the first equation the left member is expanded in powers of x and then x is replaced by $2.5\cos\theta$. Powers of the cosine are then reduced to trigonometric polynomials.

Two beautiful graphs are given of Legendre's $P_n(\cos\theta)$ $(n = 1, \cdots, 11)$, and certain associated functions $P_n^{(m)}(\cos\theta)$. Unfortunately no scales are indicated. Finally there is a discussion of the double Fourier series

$$A(x, y) = \sum_{h=0}^{n} \sum_{k=0}^{m} a_{hk} \cos 2\pi hx \cos 2\pi ky$$

used in the Patterson method for determining interatomic distances in crystals. Here the machine is set to draw a series of $n + 1$ curves

$$C_{kx} = \sum_{h=0}^{m} a_{hk} \cos 2\pi hx. \quad (k = 0, \cdots, n).$$

Then, reading 5 values of C_{kx} on each curve (for $x = 0(.02)1$), the machine is set to draw the 5 curves

$$A(x, y) = \sum_{k=0}^{n} C_{kx} \cos 2\pi ky.$$

These curves can then be used to prepare a contour map of the function $A(x, y)$.

D. H. L.

10[Z].—R. L. DIETZOLD, "The isograph a mechanical root finder," and R. O. MERCNER, "The mechanism of the isograph," *Bell Laboratories Record*, v. 16, 1937, p. 130–134 and p. 135–140.

These short articles describe the theory, practice and construction of a large instrument designed in 1928 by T. C. Fry primarily for the solution of algebraic equations of degrees ≤ 10. The theory behind the machine is briefly set forth in a review of the S. L. Brown harmonic analyzer, to which review the reader is referred (*MTAC*, p. 128–129). The mechanical methods of generating and adding the ten pairs of motions $a_n r^n \cos n\theta$, $a_n r^n \sin n\theta$ and the drawing of this sum are practically the same as those of the Brown analyzer. There is one difference however. In the latter machine the two motions of any one frequency may be set with any preassigned phase difference, while in the Isograph this difference is inexorably 90°. This means that the use of the Isograph is restricted to equations with real coefficients. An interesting feature of the Isograph is a set of ten cylindrical slide rules driven with frequencies from 1 to 10, for computing the values of $a_n r^n$ $(n = 1, \cdots, 10)$ which are the ten amplitudes to be set on the cross heads. These amplitudes must not exceed an inch and a half (as compared with over 6 inches in the Brown analyzer) so that frequent changes in scale must be required for accommodating rather different values of r.

The photographs and diagrams illustrating these articles have been put on 12 slides, and a film has been made. Since these may be borrowed from the Bell Laboratories (New York), together with mimeographed material about the machine, there is the basis for an interesting lecture about the Isograph for mathematics classes and club meetings.

D. H. L.

NOTES

11. ALFRED LODGE.—In referring to sketches of two tablemakers in *Who was Who 1929–1940*, London, 1941 (N 1) a third one was overlooked, namely: of ALFRED LODGE (1854–1937), brother of Sir Oliver Lodge (1851–1940), long principal of the University of Birmingham, and of Sir Richard Lodge (1855–1936), professor of history at the University of Edinburgh. Alfred Lodge was Fereday fellow of St. John's College, Oxford, 1876–91, professor of pure mathematics at the Royal Indian Engineering College, Cooper's Hill, 1884–1904, and assistant master (mathematics) of Charterhouse from 1904 until his retirement in 1919, at the age of 65. He was president of the Mathematical Association, 1897–98, and scores of his communications appeared in the *Mathematical Gazette*. For some years he was secretary and recorder of Section A of the B.A.A.S. He was a member of its Committee on Calculation of Mathematical Tables for nearly fifty years; except for Airey, more of his tables were published than those of any other member (see *MTAC*, p. 75). He was a joint author of the Committee's *Factor Table* (1935), of which three independent copies were made; compare N **13**. Since Lodge alone had been on the Committee from the day on which Bessel functions were first discussed up to 1937, when the Committee published its first volume of tables of *Bessel Functions*, the volume was dedicated to him. It appeared, however, a few days after his death. See also *The Times*, London, 6 Dec. 1937, p. 14; and C. O. TUCKEY, "Alfred Lodge," *Math. Gazette*, v. 22, 1938, p. 3–4.

12. INTERPOLATION.—If there is suspicion of the accuracy of an entry in a table with equal argument-intervals, the suspicious entry u_0 may be checked in terms of the six adjacent entries u_{-3}, u_{-2}, u_{-1} and u_1, u_2, u_3, by the

formula $u_0 = 0.75(u_{-1} + u_1) - 0.30(u_{-2} + u_2) + 0.05(u_{-3} + u_3) - \Delta^6 u_{-3}/20$. If sixth differences are negligible, as is often the case in published tables, the term $\Delta^6 u_{-3}/20$ is omitted, as a zero remainder-term.

In many tables an earlier difference than the sixth may be neglected, yet the formula above is to be preferred to a simpler one using less than six entries, because no division on the machine is needed. [This is because the coefficient of the central term in $(E - 1)^6$ is 20; for no other power has this coefficient powers of 2 and 5 as its only factors].

This formula is fairly obvious when pointed out, but I have never seen it in print.

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13. JOHANN THEODOR PETERS.—An obituary notice by W. R. A. Klose in *Z. angew. Math. u. Mech.*, v. 22, Apr. 1942, p. 220, states that "Jean Peters," the greatest mathematical table-maker of all time, died on 24 August 1941. Since he was born on 25 August 1869 his span of life completed almost exactly 72 years. His full name, as given above, is as he sent it for publication in Poggendorff's *Biographisch-literarisches Handwörterbuch*, v. 6, part 3, 1938. In his published mathematical tables he always gave his name as "Dr. J. Peters" or "Prof. Dr. J. Peters." In the *Porträtgallerie*, 1931 (*MTAC*, p. 12) the name is "Johannes Peters." Since we know the name which Peters himself regarded as complete for the last decade at least, whence the name "Jean"?

Peters was born in the rather genial atmosphere of Cologne, and took his doctorate in 1895 at the University of Bonn under the astronomer Friedrich Küster. According to the *Jahresverzeichnis . . . Universitäten . . . Abhandlungen*, for 1894–95, his dissertation was printed under the name "Jean Peters." Hence *Deutsches Bücherverzeichnis*, *Kürschners Gelehrten-Kalender*, *Minerva*, and other standard sources, preserved this form which was also copied on catalogue cards of the Library of Congress, and used by Klose, an associate of earlier years. From 1899 until his retirement in 1937 he was on the staff of the Astronomisches Recheninstitut of Berlin-Lichterfelde, where rigid formalism in name and otherwise was more the order of the day. Here he was the one principally responsible for the annual *Berliner Astronomisches Jahrbuch*, and *Nautisches Jahrbuch*. Besides elaborate tables for dealing with problems of pure astronomy (Kepler's and coordinate transformations, for example) he published *New Calculating Tables for multiplication and division by all numbers from one to four places* (1909), an 8-place table (with Bauschinger) of the logarithms of numbers 1–200 000 (1910), 7-, and 10-place (*Anhang* with J. Stein) tables of the logarithms of numbers 1–100 000 (1940, and 1922), *Zweiundfünfzigstellige Logarithmen* (with J. Stein, 1919), the *Factor Table giving the Complete Decomposition of all numbers less than 100 000* (B.A.A.S., *Math. Tables*, v. 5, 1935), and fifteen other volumes of tables of logarithmic and natural trigonometric functions. The first of these volumes was (with Bauschinger) the eight-place logarithmic table for each sexagesimal second (1911). This was later followed by tables of varying number of decimal places (3, 5, 6, 7, 8, 10, and 21) for logarithmic, machine, and other calculations, with

sexagesimal division of the quadrant, with decimal division of the degree, with time as argument, and with centesimal division of the quadrant. Since during 1938–40 there were three editions of his six-place table for each $0^g.001$, its possible use in connection with war problems is suggested.

Klose refers to a large number of unpublished tables by Peters, but names only one, an eight-place table of reciprocals. Earlier reference has been made to eleven others; four of these (*MTAC*, p. 39, 114) being seven-, eight-, and ten-place tables of natural trigonometric functions for the centesimal division of the quadrant, and eight-place table of addition and subtraction logarithms (see L. J. Comrie, Br. Astr. Assoc., *Handbook*, 1929, p. 39). From information at hand it seems possible that one of these, the seven-place table for each $0^g.001$, was really published at Leipzig in 1939. The seven-place logarithmic table for each $0^g.001$ appeared in 1940; see RMT **125**. Fifth and sixth unpublished mss. are referred to in RMT **78**, **127**, and five more in RMT **125**.

"J. Peters was an accurate and speedy calculator capable of sustained effort. He excelled in organizing great calculating tasks. His methods of calculation, and of avoidance of errors, were well thought through. Unfortunately these are scarcely known except to a small circle of his most intimate collaborators." The excellent appearance of type-display in his published tables is also noted.

[There are references to Peters and his works in *MTAC*, p. 7, 8, 10–15, 25, 26, 39, 42, 43, 45, 55, 57–59, 64, 65, 82–84, 86–89, 100, 114, 121, 122, 125, 143–148, 162–164, 170, 171].

R. C. A.

14. RMT 98.—It seems worth while supplementing this review of MATHEMATICAL TABLES PROJECT *Miscellaneous Physiacl Tables*, by remarking that, in Table V if we regard x as a sine, G and xG are secants and tangents respectively. Similarly if x is a cosine, G and xG are cosecants and cotangents. This purely trigonometrical relationship is not pointed out in the Introduction. It is quite evident that this table may be used for applications other than electronic functions. x (review) $= \beta$ (Table V).

L. J. C.

15. RUSSIAN EDITION OF BARLOW'S TABLES.—The third edition of *Barlow's Tables of Squares, Cubes, Square Roots, Cube Roots, and Reciprocals, of all Integer Numbers up to* 10 000, *edited by* L. J. COMRIE, appeared at London, Spon in 1930 with xii, 208 p. In *MTAC*, p. 17, we noted incidentally the pirated Russian government edition of this. It has two title-pages; the one on the left intended to be the English original, while opposite it is the following (when transliterated): *Tablit͡sy Barlou kkadratov, kubov, korneĭ kvadratnykh, korneĭ kubichnykh i obratnykh velichin t͡selykh chisel ot* 1 *do* 10 000. Gosudarstvennoe tekhniko-teoreticheskoe Izdatyelit͡sestvo, Moscow and Leningrad, 1933. xii, 208 p. There were 10 000 copies in the edition. The last words of the title on the English-title page of the Russian edition are, "all Integer Numbers up 1 to 10 000."

16. A RUSSIAN TABLE OF PRODUCTS.—To meet the needs of the great army of workers on Collective Farms, in Industry, and in Factory Kitchens, and Cooperative Stores, in its second five-year period, the Russian Government published in 1936 an edition of 10 000 copies of the following bound volume: I. P. ZOLOTAREV, *Universalnie Raschetnie Tablitsy. Prakticheskoe Posobie dlia Bukhgalterov i schetnykh rabotnikov.* [Universal Calculating Tables. Practical Help for Bookkeepers and Computing Workers] Moscow and Leningrad, 565 p. +1 p. errata. 11 × 14.5 cm. Scattered through the volume are 15 other errata slips with one or two corrections on each. By means of this table one may read off at once the products of two numbers N_1 and N_2, if $N_1 = 0.01(0.01)0.07,0.10(0.05)1(1)250$; and $N_2 = 1(1)250,255-(10)525$. Examples of practical problems solved by the use of such a table are given on p. 3–9.

QUERIES

7. BRIGGS' ARITHMETICA LOGARITHMICA.—Compare N8. Since the publication of J. W. L. Glaisher, *Report of the Committee on Mathematical Tables*, London, 1873, of J. Henderson, *Bibliotheca Tabularum Mathematicarum*, part 1, *Logarithmic Tables*, Cambridge, 1926, and of A. Fletcher's letter in *Nature*, v. 148, 1941, p. 728 (to which my attention was drawn by L. J. Comrie), it has been generally known that there were some copies of Henry Briggs, *Arithmetica Logarithmica*, 1624, 388 p., with an extra 12 pages containing the logarithms of numbers 100,001(1)101,000, to 14D, and the square roots of integers 1(1)200, to 11D, with first differences in each case. There are copies of such enlarged editions in the library of Trinity College, Cambridge, and in the Savile collection of the Bodleian Library, Oxford. Glaisher tells us that in 1873 C. W. Merrifield [1] owned such a copy. Mr. Fletcher reported that the Harold Cohen Library of the University of Liverpool had a copy of just the 12 supplementary pages of the work of Briggs. Where are other copies of the *Arithmetica Logarithmica*, with the extra pages described above?

R. C. A.

[1] The writer has many of Merrifield's letters to Glaisher, 1873–81, mostly dealing with questions about tables.

QUERIES—REPLIES

7. AN ENGEL TABLE (Q6).—"I am the owner of a copy of this table of natural trigonometric functions, containing 95 pages (19.7 × 29.9 cm.). The volume was an outcome of a meeting of German, Austrian, and Hungarian geodesists at Berlin in Nov. 1917, when the decimal division of the non-agesimal degree was agreed upon as desirable for tables. The Preussische Landesaufnahme undertook the preparation of the three volumes of logarithmic trigonometric tables necessary for this purpose, and published 10-place (1919), 7-place (1921), 6-place (1921) tables, each for sine, tangent, cotangent, cosine, for every thousandth of a degree. These volumes[1] were all by the remarkable J. T. PETERS. As a contribution from Austria to machine calculation ENGEL initiated the computation of the tables published in 1920, under the title given in the Query. It is a 10-place table for

sine, cosine and tangent, for every hundredth of a degree, with first differences. But additional tabular material makes possible the ready determination of the values of the sine and cosine for each thousandth of a degree in the quadrant, and of the tangent for each thousandth of a degree 0° to 45°. Errors may occur in the values of sine and cosine up to 0.3 of a unit in the tenth decimal and up to 2.5 units for the tangent. For more exact computations, a table of the interpolation coefficients for the second differences, progressing by 0.01, is added."—This information was received by the EDITORS in a letter, dated 5 November, 1943, from L. W. POLLAK (66 Hollybrook Road, Clontarf, Dublin, Ireland), formerly professor of geophysics and director of the Meteorological Observatory at the German University in Prague.

[1] To meet the needs of an optical establishment Peters had earlier prepared a 7-place table of these trigonometric functions for every thousandth of a degree; this was published in 1918 (see RMT 79). For a number of years thereafter, however, this volume was not readily accessible to the public. Then later (1935) came E. Buckingham's 8-place table for every hundredth of a degree (see MTE 12).—EDITORS.

CORRIGENDA

P. 4, 8 (2), 52, 56, *for* TALLQUIST, *read* TALLQVIST. P. 22, l. 4 from bottom, *for* 20, *read* 16.5. P. 36, l. 14 *for* For, *read* "For. P. 39, l. 8, *for* decimal, *read* decimals. P. 46, l. 13 from bottom, *read* Sakamoto. P. 52, l. 6 from bottom, *read* $f\left(\frac{p+q}{2}+\frac{q-p}{2}u\right)$. P. 53, l. 3, supply the factor $|f^{(2n)}(\xi)|$ to the third member of (3). P. 56, l. 10, *read* J. C. Adams. P. 69, l. 12, *for* any Tables, *read* any other Tables. P. 70, no. 13, l. 2, *for* 8D", *read* eight places [should be 6D]"; l. 5, *for* bei x, *read* bei' x; *for* are the 41, not "35 items", *read* include the 36, not "35 entries". P. 71, *1916*, 23, l. 2, *substitute* n for each x. P. 72, 31, l. 3, *for* 5D or 6D, *read* 5S or 6S; 38, l. 2, *for* A. N. Lowan, *read* R. v. MISES.

P. 78, l. 1, *for* 101 [J, L], *read* 101 [J, K, L]; *for* percentage of, *read* percentage points of. P. 78, l. 30, *read for Biological, Agricultural and*. P. 85, l. 5, *read* C. W. Merrifield; l. 13 from bottom, *read* J. de MENDIZÁBAL TAMBORREL. P. 86, in MTE 9, *delete* the "erratum" on p. 8, l. 10, reported by GERTRUDE M. COX. P. 89, first table, *for* 21, *read* 25. P. 96, footnote 2, *for Parfait, read Parfaits*. P. 97, N 8, l. 5, *for* 1560–61, *read* 1560/61. P. 108, 3A, *for* 1941, *read* 1940. P. 113, l. 15, *for* 15D, *read* 16D. P. 121, l. 15, *for* 17° 39′ 50″, *read* 17° 39′ 40″.

CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
Professor DAVIS, *chairman*, Professor ELDER
Professor KETCHUM, Professor LOWAN

F. Theory of Numbers
Professor LEHMER

G. Higher Algebra
Professor LEHMER

H. Numerical Solution of Equations
J. Summation of Series

I. Finite Differences. Interpolation
K. Statistics
Professor WILKS, *chairman*, Professor COCHRAN, Professor CRAIG
Professor EISENHART, Doctor SHEWHART

L. Higher Mathematical Functions
M. Integrals
Professor BATEMAN

N. Interest and Investment
O. Actuarial Science
Mister ELSTON, *chairman*, Mister THOMPSON, Mister WILLIAMSON

P. Engineering

Q. Astronomy
Doctor ECKERT, *chairman*, Doctor GOLDBERG, Miss KRAMPE

R. Geodesy
S. Physics
T. Chemistry
U. Navigation

Z. Calculating Machines and Mechanical Computation
Doctor COMRIE, *chairman*, Professor CALDWELL, *vice-chairman*
Professor LEHMER, Doctor MILLER, Doctor STIBITZ, Professor TRAVIS

EDITORIAL NOTICES

The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the executive committee.

Either the next number of *MTAC*, or the one following it, will probably be a greatly enlarged special issue, devoted to a *Guide to Tables in Bessel Functions*.

Volume I is to contain at least 8 numbers continuously paged.

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